

# HODGE GROUPS OF HODGE STRUCTURES WITH HODGE NUMBERS

$$(n, 0, \dots, 0, n)$$

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**ABSTRACT.** This paper studies the possible Hodge groups of simple polarizable  $\mathbb{Q}$ -Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$ . In particular, it generalizes earlier work of Ribet and Moonen-Zarhin to completely determine the possible Hodge groups of such Hodge structures when  $n$  is equal to 1, 4, or a prime  $p$ . In addition, the paper determines possible Hodge groups, under certain conditions on the endomorphism algebra, when  $n = 2p$ , for  $p$  an odd prime. A consequence of these results is that both the Hodge and General Hodge Conjectures hold for all powers of a simple  $2p$ -dimensional abelian variety satisfying the aforementioned conditions.

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## 1. INTRODUCTION

A  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$  is a  $\mathbb{Q}$ -vector space  $V$  together with a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = V^{w,0} \oplus V^{0,w}$$

such that the subspaces  $V^{w,0}$  and  $V^{0,w}$  are conjugate to each other. Recently, Totaro [27, Theorem 3.1] classified all of the possible endomorphism algebras of polarizable  $\mathbb{Q}$ -Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$ .

Complex abelian varieties of dimension  $n$  are equivalent, up to isogeny, to polarizable  $\mathbb{Q}$ -Hodge structures of weight 1 with Hodge numbers  $(n, n)$ . Thus, Totaro's result generalizes a result of Shimura [25, Theorem 5], which classifies all the possible endomorphism algebras of a complex abelian variety of fixed dimension.

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If  $V$  is an even-weight  $\mathbb{Q}$ -Hodge structure with decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=w} V^{p,q},$$

then the Hodge classes of  $V$  are the elements purely of type  $(p, p)$  in this decomposition. These Hodge classes are the subject of the long-standing Hodge Conjecture. The Hodge Conjecture states that in the cohomology of a smooth complex projective variety  $X$ , the Hodge classes of type  $(p, p)$  are  $\mathbb{Q}$ -linear combinations of cohomology classes of algebraic subvarieties of codimension  $p$  in  $X$ .

For any rational sub-Hodge structure  $W \subset V$ , the *level* of  $W$  is  $l(W) = \max\{p - q \mid W^{p,q} \neq 0\}$ . Then the General Hodge Conjecture states that for a rational substructure  $W \subset H^w(X, \mathbb{Q})$  such that  $l(W) = w - 2p$ , there exists a Zariski-closed subset  $Z$  of codimension  $p$  in  $X$  such that  $W$  is contained in  $\ker(H^w(X, \mathbb{Q}) \rightarrow H^w(X - Z, \mathbb{Q}))$ .

The *Hodge group* of a Hodge structure  $V$  is a connected algebraic  $\mathbb{Q}$ -subgroup of  $SL(V)$  whose invariants in the tensor algebra generated by  $V$  and its dual  $V^*$  are exactly the Hodge classes. Thus Hodge groups of  $\mathbb{Q}$ -Hodge structures are objects of great interest towards a better understanding of both the Hodge and General Hodge Conjectures.

In this paper, we use Totaro's results about the endomorphism algebras of polarizable Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$  in order to characterize the possible Hodge groups of the simple such Hodge structures. In particular, in Proposition 7.1, Theorem 7.2, and Theorem 7.3, we completely determine the possible Hodge groups when  $n$  is equal to 1, a prime  $p$ , or 4, respectively. Moreover, when  $n = 2p$ , for  $p$  an odd prime, Theorem 7.7 determines the possible Hodge groups of such Hodge structures, under certain conditions on the endomorphism algebra of  $V$  under which we may consider  $V$  as a Hodge structure over an imaginary quadratic field. Observe, for instance, that  $n = 1, 4, p$ , and  $2p$  covers cases for all  $n < 8$ .

The results for  $n = 1, p$ , and 4 generalize known results about the possible Hodge groups of simple complex abelian varieties, while the results for  $n = 2p$  are new.

In particular, Ribet showed in [21] that the Hodge group of a simple complex abelian variety  $X$  of prime dimension is always equal to the Lefschetz group of  $X$ , meaning, roughly speaking, that the Hodge group of  $X$  is always as large as possible. We show that, in fact, the above holds for all simple polarizable Hodge structures with Hodge numbers  $(p, 0, \dots, 0, p)$ .

Moonen and Zarhin addressed the case of simple complex abelian fourfolds in [14]. They showed that the Hodge group of a simple complex abelian fourfold  $X$  is equal to the Lefschetz group of  $X$  except in the cases when the endomorphism algebra of  $X$  is  $\mathbb{Q}$  or a CM field of degree 2 or 8 over  $\mathbb{Q}$ . In all of these exceptional cases, an additional group is also possible.

We show similarly that for a simple polarizable  $\mathbb{Q}$ -Hodge structure  $V$  with Hodge numbers  $(4, 0, \dots, 0, 4)$ , the Hodge group is equal to the Lefschetz group of  $V$  except in the cases when the endomorphism algebra of  $V$  is  $\mathbb{Q}$  or a CM field of degree 2 or 8 over  $\mathbb{Q}$ . When the endomorphism algebra of  $V$  is a CM field, the additional possible group is analogous to the one found by Moonen and Zarhin. In the case when the endomorphism algebra of  $V$  is equal to  $\mathbb{Q}$ , Moonen and Zarhin used a construction of Mumford's [16, Section 4] to show that the group  $SL(2) \times SO(4) \cong SL(2)^3$  is also realizable as the Hodge group of a simple abelian fourfold. We show that the analogous group  $SL(2) \times Sp(4)$  for an even-weight simple polarizable Hodge structure with Hodge numbers  $(4, 0, \dots, 0, 4)$  does not occur. However we show that a different non-Lefschetz group, namely the group  $SO(7)$ , is possible as the Hodge group of an even-weight simple polarizable Hodge structure with Hodge numbers  $(4, 0, \dots, 0, 4)$  and endomorphism algebra equal to  $\mathbb{Q}$ .

The results of this paper about the possible Hodge groups of simple polarizable Hodge structures with Hodge numbers  $(2p, 0, \dots, 0, 2p)$  have implications in terms of the Hodge Conjecture and General Hodge Conjecture for simple  $2p$ -dimensional abelian varieties. In particular, for  $X$  a simple  $2p$ -dimensional abelian variety satisfying certain conditions, Corollary 8.1 shows that the validity of the Hodge Conjecture for all powers of  $X$  implies the validity of the General Hodge

Conjecture for all powers of  $X$ . Corollary 8.4 then shows that for such varieties, the Hodge Ring

$$\mathcal{B}^\bullet(X^k) = \bigoplus_{l \geq 0} \left( H^{2l}(X^k, \mathbb{Q}) \cap H^{l,l} \right)$$

is generated by divisors. It follows that the Hodge Conjecture and thus, by Corollary 8.1 also the General Hodge Conjecture, holds for all powers of  $X$ .

In order to motivate the study of Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$ , let us say that a Hodge structure *comes from geometry* if it is a summand of the cohomology of a smooth complex projective variety defined by a correspondence. Griffiths transversality implies that a variation of Hodge structures of weight at least 2 with no two adjacent non-zero Hodge numbers is locally constant [28, Theorem 10.2]. In particular, this applies to Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$ . It follows that only countably many Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$  and weight at least 2 can come from geometry.

Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$  arise naturally in geometry, of course, as abelian varieties or, for instance, as rigid Calabi-Yau threefolds. Moreover, we can produce a Hodge structure with Hodge numbers  $(n, 0, n)$  as follows. If  $X$  is a smooth complex projective surface with maximal Picard number, namely the Picard number of  $X$  is equal to  $h^{1,1}(X)$ , then  $H^2(X, \mathbb{Q})$  modulo the subspace of Hodge classes is a Hodge structure with Hodge numbers  $(p_g(X), 0, p_g(X))$ . For a survey of surfaces with maximal Picard number, see Beauville's recent work [5].

In [23], Schreieder shows how to construct a smooth complex projective variety with a prescribed set of Hodge numbers in a given degree  $w$  (under suitable conditions on  $h^{p,p}$  when  $w$  is even). This gives yet another way to produce a Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  geometrically. Additionally, Arapura's work in [4] shows how to construct a Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  as a direct summand of the cohomology of a power  $E^N$  of a CM elliptic curve. More generally, however, very little is known about the subset of the period domain of all Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$  consisting of those coming from geometry.

The organization of the paper is as follows. In Section 2, we introduce the necessary background and notation. In particular, we discuss how the endomorphism algebra of a Hodge structure of the relevant type must be of Type I, II, III, or IV in Albert's classification of division algebras over a number field that have positive involution [3, Chapter X, §11]. Section 3 then gives results about possible Hodge groups of simple polarizable Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$  having endomorphism algebra  $L$  of Type I, subject to various conditions on the relationship between  $n$  and  $[L : \mathbb{Q}]$ . Section 4 provides analogous results for  $L$  of Type II or III. Section 5 analyzes the situation when  $L$  is of Type IV. This is the hardest case to characterize and thus Section 5 primarily focuses on Hodge structures whose endomorphism algebra  $L$  satisfies the additional hypothesis that  $V$  may be viewed as a Hodge structure over some CM field  $E$  with an embedding into  $L$ . Section 6 generalizes results of Ribet [20] and Orr [19, Theorem 1.1] to give a lower bound on the dimension of the Hodge group for Hodge structures of the type considered.

The main results of the paper occur in Section 7, where the possible Hodge groups for  $n = 1, 4, p$ , and  $2p$  are characterized. Section 8 then gives corollaries of the results in Section 7 in the context of the Hodge Conjecture and General Hodge Conjecture for simple abelian varieties of dimension  $2p$ .

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## 2. NOTATION

A  $\mathbb{Q}$ -Hodge structure  $V$  is a finite dimensional  $\mathbb{Q}$ -vector space together with a decomposition into linear subspaces

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{(p,q) \in \mathbb{Z}^2} V^{p,q},$$

such that  $\overline{V^{p,q}} = V^{q,p}$  and such that the *weight grading*  $p+q$ , is defined over  $\mathbb{Q}$ . A Hodge structure  $V$  is of *pure weight*  $w$  if  $V^{p,q} = 0$  whenever  $p+q$  is not equal to  $w$ . For example, any smooth complex projective variety  $X$  has a  $\mathbb{Q}$ -Hodge structure of (pure) weight  $w$  on  $H^w(X, \mathbb{Q})$ .

Alternatively, a  $\mathbb{Q}$ -Hodge structure can be defined as a finite dimensional  $\mathbb{Q}$ -vector space  $V$  together with a homomorphism of  $\mathbb{R}$ -algebraic groups

$$h : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \rightarrow GL(V \otimes_{\mathbb{Q}} \mathbb{R}).$$

Here  $h(z)$  acts on  $V^{p,q}$  as multiplication by  $z^{-p}\bar{z}^{-q}$ .

A  $\mathbb{Q}$ -Hodge structure  $V$  has *Hodge numbers*  $(a_0, a_1, \dots, a_w)$  if  $V$  has weight  $w$  and:

$$\dim_{\mathbb{C}} V^{i,w-i} = \begin{cases} a_i & \text{for } 0 \leq i \leq w \\ 0 & \text{otherwise} \end{cases}$$

A *polarization* of a  $\mathbb{Q}$ -Hodge structure  $V$  of weight  $w$  is a bilinear form  $\langle, \rangle : V \times V \rightarrow \mathbb{Q}$  that is alternating if  $w$  is odd, symmetric if  $w$  is even, and whose extension to  $V \otimes_{\mathbb{Q}} \mathbb{C}$  satisfies:

- (1)  $\langle V^{p,q}, V^{p',q'} \rangle = 0$  if  $p' \neq w-p$
- (2)  $i^{p-q}(-1)^{\frac{w(w-1)}{2}} \langle x, \bar{x} \rangle > 0$  for all nonzero  $x \in V^{p,q}$

Note, for instance, that for  $X$  a smooth complex projective variety, a choice of ample line bundle on  $X$  will determine a polarization on the  $\mathbb{Q}$ -Hodge structure  $H^w(X, \mathbb{Q})$ . The category of polarizable  $\mathbb{Q}$ -Hodge structures is a semisimple abelian category [15, Theorem 1.16]. All Hodge structures considered in this paper will be polarizable.

The *Mumford-Tate group*  $MT(V)$  of a  $\mathbb{Q}$ -Hodge structure  $V$  is the  $\mathbb{Q}$ -Zariski closure of the homomorphism

$$h : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \rightarrow GL(V \otimes_{\mathbb{Q}} \mathbb{R}).$$

Define the cocharacter

$$\mu : \mathbb{G}_m \rightarrow R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$$

to be the unique cocharacter such that  $z \circ \mu$  is the identity in  $\text{End}(\mathbb{G}_m)$  and  $\bar{z} \circ \mu$  is trivial. Then the Mumford-Tate group of  $V$  may be alternatively described as the smallest  $\mathbb{Q}$ -algebraic group contained in  $GL(V)$  such that

$$h \circ \mu : \mathbb{G}_m \rightarrow GL(V \otimes_{\mathbb{Q}} \mathbb{C})$$

factors through  $MT(V)_{\mathbb{C}}$ .

Instead of working with the Mumford-Tate group, we will generally work with a slightly smaller group, called the *Hodge group*  $Hg(V)$  of  $V$ . The Hodge group is the  $\mathbb{Q}$ -Zariski closure of the restriction of the homomorphism  $h$  to the circle group

$$U_1 = \ker(\text{Norm} : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \rightarrow \mathbb{G}_m).$$

If the Hodge structure  $V$  is pure of weight 0, then  $Hg(V)$  and  $MT(V)$  coincide. If  $V$  is pure of nonzero weight, then  $MT(V)$  contains  $\mathbb{G}_m$ , and in fact, is equal to the almost direct product in  $GL(V)$ :

$$\mathbb{G}_m \cdot Hg(V).$$

If  $V$  is a polarizable  $\mathbb{Q}$ -Hodge structure, then both the Mumford-Tate group and the Hodge group of  $V$  are connected, reductive algebraic groups over  $\mathbb{Q}$ . In essence, both the Mumford-Tate group and the Hodge group of  $V$  encode the complexity of the Hodge structure  $V$ .

Let  $E$  be a number field. Then define an *E-Hodge structure* to be a  $\mathbb{Q}$ -Hodge structure  $V$  together with a homomorphism of  $\mathbb{Q}$ -algebras:

$$E \rightarrow \text{End}_{\mathbb{Q}-HS}(V).$$

Suppose  $E$  is, in fact, a totally real or CM field, where a *CM field* means a totally imaginary quadratic extension of a totally real number field. Then, writing  $a \rightarrow \bar{a}$  for the involution on  $E$  given by complex conjugation (which is the identity involution if  $E$  is totally real), we may define a

*polarized E-Hodge structure* to be a polarized  $\mathbb{Q}$ -Hodge structure together with a homomorphism  $E \rightarrow \text{End}_{\mathbb{Q}\text{-HS}}(V)$  of  $\mathbb{Q}$ -algebras with involution. Namely, the form  $\langle, \rangle : V \times V \rightarrow \mathbb{Q}$  satisfies

$$\langle ax, y \rangle = \langle x, \bar{a}y \rangle$$

for all  $a \in E$  and all  $x, y \in V$ . In fact, if  $V$  is an  $E$ -Hodge structure whose underlying  $\mathbb{Q}$ -Hodge structure is polarizable, then  $V$  is polarizable as an  $E$ -Hodge structure [27, Lemma 2.1]. There does not seem to be a good definition of a polarized  $E$ -Hodge structure for  $E$  a number field which is not totally real or a CM field.

Let  $E$  be a totally real or CM field with  $[E : \mathbb{Q}] = r$ . Let  $\Sigma(E)$  be the set of embedding of  $E$  into  $\mathbb{C}$  and let  $\sigma_1, \dots, \sigma_r$  be the elements of  $\Sigma(E)$ . If  $V$  is an  $E$ -Hodge structure, then each  $V^{p,q}$  in the decomposition of  $V \otimes_{\mathbb{Q}} \mathbb{C}$  is an  $E \otimes_{\mathbb{Q}} \mathbb{C}$ -module. We have:

$$E \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\sigma_i \in \Sigma(E)} \mathbb{C}.$$

So then each  $V^{p,q}$  splits as a direct sum:

$$V^{p,q} = \bigoplus_{\sigma_i \in \Sigma(E)} V^{p,q}(\sigma_i),$$

where  $V^{p,q}(\sigma_i)$  is the subspace of  $V \otimes_{\mathbb{Q}} \mathbb{C}$  where  $E$  acts via  $\sigma_i$ .

We then say that  $V$  has *Hodge numbers*  $(a_0, \dots, a_w)$  if, for each embedding  $\sigma_i \in \Sigma(E)$ , the summand  $V^{j,w-j}(\sigma_i)$  has complex dimension  $a_j$  for all  $j$ . Note that if  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(a_0, \dots, a_w)$ , then  $V$  is a  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(ra_0, \dots, ra_w)$ .

*Remark 2.1.* If  $V$  is a polarizable  $\mathbb{Q}$ -Hodge structure with Hodge group  $\{1\}$ , then  $h(z)$  acts as the identity on nonzero  $V^{p,q}$  in the decomposition of  $V \otimes_{\mathbb{Q}} \mathbb{C}$ . However, by definition  $h(z)$  acts on  $V^{p,q}$  as multiplication by  $z^{-p}\bar{z}^{-q}$ . Hence, if  $Hg(V) = \{1\}$ , then  $V^{p,q}$  is zero for all  $p, q$  such that  $p \neq q$ .

Let  $V$  be a simple polarized  $\mathbb{Q}$ -Hodge structure. Then the endomorphism algebra  $L = \text{End}_{\mathbb{Q}\text{-HS}}(V)$  of  $V$  is a division algebra over  $\mathbb{Q}$  with an involution  $a \rightarrow \bar{a}$ , given by  $\langle ax, y \rangle = \langle x, \bar{a}y \rangle$ . This involution is called the *Rosati involution*.

The Rosati involution is a *positive involution*, meaning that, for every  $\sigma_i \in \Sigma(L)$ , the reduced trace  $\sigma_i(\text{tr}_{\mathbb{Q}}^L(x\bar{x}))$  is positive as an element of  $\mathbb{R}$  for all nonzero  $x$  in  $V$  [15, Remark 1.20]. It follows that if  $L$  is a field, then  $L$  is either a totally real or a CM field and hence the Rosati involution on  $L$  just corresponds to complex conjugation.

Now, suppose  $L$  is the endomorphism algebra of a simple polarized  $\mathbb{Q}$ -Hodge structure. Let  $F_0$  be the center of  $L$  and let  $F$  be the subfield of  $F_0$  fixed by the Rosati involution. Then Albert's classification of division algebras over a number field that have positive involution [3, Chapter X, §11] yields that  $L$  must one of the following four types:

Type I:  $L = F$  is totally real

Type II:  $L$  is a totally indefinite quaternion algebra over the totally real field  $F$

Type III:  $L$  is a totally definite quaternion algebra over the totally real field  $F$

Type IV:  $L$  is a central simple algebra over the CM field  $F_0$

*Remark 2.2.* If the endomorphism algebra  $L$  of a polarizable  $\mathbb{Q}$ -Hodge structure  $V$  has no simple factors of Type IV, then the Hodge group of  $V$  is semisimple. [15, Proposition 1.24]

The elements purely of type  $(p, p)$  in the Hodge decomposition of a  $\mathbb{Q}$ -Hodge structure  $V$  are called *Hodge classes*. The endomorphism algebra  $L$  of  $V$  can be viewed as the set of Hodge classes of the Hodge structure on  $\text{End}_{\mathbb{Q}}(V)$ . Hence, these Hodge classes are the elements of  $\text{End}_{\mathbb{Q}}(V)$  which are invariant under the action of the Hodge group  $Hg(V)$ . Namely we have:

$$L = [\text{End}_{\mathbb{Q}}(V)]^{Hg(V)}.$$

In particular, the Hodge group of  $V$  is contained in the connected component of the centralizer of  $L$  in the  $\mathbb{Q}$ -group  $SL(V)$ .

Define the *Lefschetz group*  $Lef(V)$  of  $V$  to be the connected component of the centralizer of  $L$  in:

$$\begin{cases} Sp(V) & \text{if } V \text{ is of odd weight} \\ SO(V) & \text{if } V \text{ is of even weight.} \end{cases}$$

It should be noted that the definition of the Lefschetz group used here corresponds with the definition used by Murty in [17, Section 2], which is the connected component of the identity in the definition used by Murty in [18, Section 3.6.2]. If  $\langle, \rangle$  is a polarization on  $V$ , then  $Hg(V)$  preserves  $\langle, \rangle$ . Thus, in fact, we have:

$$Hg(V) \subseteq Lef(V).$$

Now, let  $A$  be any central simple algebra over a field  $K_0$  with involution  $^-$ . Let  $K$  be the subfield of  $K_0$  fixed by  $^-$ . Define:

$$\begin{aligned} \text{Sym}(A, ^-) &= \{f \in A \mid \overline{f} = f\} \\ \text{Alt}(A, ^-) &= \{f \in A \mid \overline{f} = -f\} \end{aligned}$$

Then, as in the Book of Involutions, if  $[A : K_0] = q^2$ , we say that the involution  $^-$  on  $A$  is *orthogonal* if  $K_0 = K$  and  $\dim_K \text{Sym}(A, ^-) = \frac{q(q+1)}{2}$ . We say the involution is *symplectic* if  $K_0 = K$  and  $\dim_K \text{Sym}(A, ^-) = \frac{q(q-1)}{2}$ . Finally, we say the involution is *unitary* if  $K_0 \neq K$  [11, Proposition I.2.6]. There are no other possibilities for the involution  $^-$ .

Now, continuing to follow [11], define the group of *isometries* of  $A$  by:

$$\text{Iso}(A, ^-) = \{g \in A^* \mid \overline{g} = g^{-1}\}.$$

Then write:

$$\text{Iso}(A, ^-) = \begin{cases} O(A, ^-) & \text{if } ^- \text{ is orthogonal} \\ Sp(A, ^-) & \text{if } ^- \text{ is symplectic} \\ U(A, ^-) & \text{if } ^- \text{ is unitary} \end{cases}$$

If the involution  $^-$  is orthogonal, then, as an algebraic group, the kernel  $O^+(A, ^-)$  of the reduced norm map  $O(A, ^-) \rightarrow \{\pm 1\}$  is a  $K$ -form of  $SO(q)$ , meaning the two groups are isomorphic over an algebraic closure of  $K$ . If the involution  $^-$  is symplectic, then  $Sp(A, ^-)$  is a  $K$ -form of  $Sp(q)$  (in this case  $q$  must be even). If  $^-$  is unitary, then  $U(A, ^-)$  is a  $K$ -form of  $GL(q)$ .

Suppose  $\mathfrak{g}$  is a semisimple Lie algebra over an algebraically closed field  $K$  and let  $\mathfrak{h}$  be a Cartan subalgebra of  $\mathfrak{g}$ . Let  $R$  be the root system of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$  and let  $B = \{\alpha_1, \dots, \alpha_l\}$  be a set of simple roots of  $R$  with corresponding set of coroots  $B^\vee = \{\alpha^\vee \mid \alpha \in B\}$ . If  $w_0$  is the longest element of the Weyl group of  $R$  with respect to the basis  $B$ , let  $\lambda \rightarrow \lambda' := -w_0(\lambda)$  denote the opposition involution on  $\mathfrak{h}^*$ . If  $\lambda$  is a dominant weight, we can write

$$\lambda = \sum_{\alpha \in B} c_\alpha \cdot \alpha$$

where all the  $c_\alpha$  are in  $\mathbb{Q}_{\geq 0}$ . By Lemma 3.3 in [15], we have that  $c_\alpha + c_{\alpha'}$  lies in  $\mathbb{Z}_{\geq 0}$  for all  $\alpha \in B$ . Then define

$$\text{length}(\lambda) = \min_{\alpha \in B} c_\alpha + c_{\alpha'}$$

If  $R$  is an irreducible root system, we say that a dominant weight  $\lambda$  is *minuscule* if  $\langle \lambda, \alpha^\vee \rangle$  lies in  $\{-1, 0, 1\}$  for all  $\alpha \in R$  [7, Chapter VIII, §7.3]. Then  $\text{length}(\lambda)$  is equal to 1 if and only if  $\lambda$  is a minuscule weight and  $R$  is of classical type, meaning of type  $A_l$ ,  $B_l$ ,  $C_l$ , or  $D_l$  [15, Example 3.6].

We make extensive use of Table 1, reproduced from [15]. The table describes for a given root system with minuscule weight  $\lambda$ , the corresponding representation  $V(\lambda)$  (omitting the weight 0). The table gives the dimension and autoduality of the representation  $V(\lambda)$ . Note that in the table, the

TABLE 1. Minuscale weights in irreducible root systems

Root system	Minuscale weight	Representation	Dimension	Autoduality
$A_l$	$\bar{\omega}_j (1 \leq j \leq l)$	$\wedge^j(\text{Standard})$	$\binom{l+1}{j}$	$(-1)^j$ if $l = 2j - 1$ 0 otherwise
$B_l$	$\bar{\omega}_l$	Spin	$2^l$	+ if $l \equiv 0, 3 \pmod{4}$ - if $l \equiv 1, 2 \pmod{4}$
$C_l$	$\bar{\omega}_1$	Standard	$2l$	-
$D_l$	$\bar{\omega}_1$ $\bar{\omega}_{l-1}, \bar{\omega}_l$	Standard Spin <sup>-</sup> , resp. Spin <sup>+</sup>	$2l$ $2^{l-1}$	+ + if $l \equiv 0 \pmod{4}$ - if $l \equiv 2 \pmod{4}$ 0 if $l \equiv 1 \pmod{2}$
$E_6$	$\bar{\omega}_1$ $\bar{\omega}_6$		27 27	0 0
$E_7$	$\bar{\omega}_7$		56	-1

symbol - denotes a symplectic representation, the symbol + denotes an orthogonal representation, and 0 denotes a non-self-dual representation.

### 3. TYPE I

Our goal in this section is to say something about the possible Hodge groups of simple polarizable  $\mathbb{Q}$ -Hodge structures having Hodge numbers  $(n, 0, \dots, 0, n)$  and endomorphism algebra of Type I in Albert's classification of division algebras with positive involution over a number field [3, Chapter X, §11].

*Remark 3.1.* There is an equivalence of categories between the category of  $\mathbb{Q}$ -Hodge structures of weight  $w$  and Hodge numbers  $(n, 0, \dots, 0, n)$ , and the category of  $\mathbb{Q}$ -Hodge structures of weight 1 and Hodge numbers  $(n, n)$ . This equivalence is given simply by identifying  $V^{w,0} \subset V \otimes_{\mathbb{Q}} \mathbb{C}$  with  $V^{1,0}$ . When  $w$  is odd, this equivalence preserves polarizability. However, this is not the case when  $w$  is even.

Thus when  $V$  has odd weight, its Hodge group is the same as that of the simple abelian variety of dimension  $n$  described by the above equivalence. Hence, in the odd-weight case, in order to determine the Hodge groups of  $\mathbb{Q}$ -Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$ , we may apply known results, for instance those in [15] and [21], about Hodge groups of certain abelian varieties.

**Proposition 3.2.** *Suppose  $V$  is a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$ , having endomorphism algebra  $L$  a totally real number field such that  $l = \frac{n}{[L:\mathbb{Q}]}$  is odd. Then,*

$$Hg(V) = \begin{cases} R_{L/\mathbb{Q}} Sp(2n) & \text{if } w \text{ is odd} \\ R_{L/\mathbb{Q}} SO(LV) \text{ or } R_{L/\mathbb{Q}} SU(2^k) \text{ (for } k \geq 3 \text{ and } 2l = \binom{2^k}{2^{k-1}}) & \text{if } w \text{ is even.} \end{cases}$$

*If  $w$  is even, then both possible groups occur. Additionally, if  $w$  is even, then we always have  $l \geq 3$ .*

*Proof.* When  $V$  is of odd weight, the result follows using the equivalence of Remark 3.1 together with a result of Ribet's [21, Theorem 1]. Thus we may assume that  $V$  is of even weight.

The restriction  $l \neq 1$  when  $w$  is even follows from Totaro's classification of the possible endomorphism algebras of  $\mathbb{Q}$ -Hodge structures of the specified type [27, Theorem 3.1].

Let  $H = Hg(V)$  be the Hodge group of  $V$ . In the case when  $L$  is totally real, the Lefschetz group of  $V$  is

$$Lef(V) = R_{L/\mathbb{Q}}SO(LV),$$

so we have:

$$H \subseteq R_{L/\mathbb{Q}}SO(LV).$$

Moreover, by Remark 2.2, because  $L$  is of Type I in Albert's classification, the group  $H$  is semisimple. So we may write  $H_{\mathbb{C}}$  as the almost direct product of its simple factors:

$$H_{\mathbb{C}} = H_1 \cdots H_s.$$

Let  $\Sigma(L)$  be the set of embeddings of  $L$  into  $\mathbb{C}$  and let  $\sigma_1, \dots, \sigma_r$  be the elements of  $\Sigma(L)$ . Then the identical representation  $V \otimes_{\mathbb{Q}} \mathbb{C}$  of  $H_{\mathbb{C}}$  decomposes as:

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma_i \in \Sigma(L)} X_{\sigma_i},$$

where the  $\mathbb{C}$ -vector spaces  $X_{\sigma_i}$  are irreducible, orthogonal representations of  $H_{\mathbb{C}}$  of dimension  $2l$ . Each  $X_{\sigma_i}$  then decomposes as a tensor product:

$$X_{\sigma_i} = \bigotimes_{j=1}^s X_{\sigma_i, j},$$

where each  $X_{\sigma_i, j}$  is a representation of the factor  $H_j$ . Some of these representations  $X_{\sigma_i, j}$  will be trivial. If the representation  $X_{\sigma_i, j}$  is nontrivial, then it is a self-dual representation, meaning the representation is either symplectic or orthogonal. Additionally, since the total representation  $X_{\sigma_i}$  is orthogonal, the number of  $j$  such that  $X_{\sigma_i, j}$  is symplectic must be even. Furthermore, since  $\dim_{\mathbb{Q}} X_{\sigma_i} = 2l$  with  $l$  odd, none of the dimensions of the  $X_{\sigma_i, j}$  can be divisible by 4.

A result of Moonen [15, Theorem 3.11] shows that all highest weights of the representations  $X_{\sigma_i, j}$  have length less than or equal to 1. Namely, the highest weights occurring are all minuscule and all simple factors  $H_i$  of  $Hg(V)_{\mathbb{C}}$  are of classical type [15, Example 3.6].

Making use of Table 1 then yields the following two facts about a simple classical Lie algebra  $\mathfrak{g}$  with minuscule weight  $\lambda$ :

- (1) If  $V(\lambda)$  is self dual, then  $V(\lambda)$  is even-dimensional
- (2) If  $V(\lambda)$  is orthogonal with  $\dim(V(\lambda)) \equiv 2 \pmod{4}$ , then  $\mathfrak{g}$  is of type either:
  - (a)  $D_m$ , where  $m \geq 1$  is odd, and the representation is the standard representation of  $\mathfrak{so}_{2m}$
  - (b)  $A_{2^k-1}$ , where  $k \geq 3$ , and the representation is  $\bigwedge^{2^{k-1}}$  (Standard)

Note that Fact 2b makes use of the combinatorial fact that, for any  $z \geq 0$ , the binomial coefficient  $\binom{2z}{z}$  is congruent to 2 mod 4 if and only if  $z$  is a power of 2.

Now since  $\dim_{\mathbb{Q}} X_{\sigma_i} = 2l$  with  $l$  odd, using Fact 1, there can only be one  $j$  such that  $X_{\sigma_i, j}$  is nontrivial. Moreover, for this  $j$ , the representation  $X_{\sigma_i, j}$  must be orthogonal, since the number of symplectic factors in the decomposition of  $X_{\sigma_i}$  must be even. Thus, applying Fact 2, the image of  $H_{\mathbb{C}}$  in  $GL_{X_{\sigma_i}}$  is  $SO_{X_{\sigma_i}}$  or, in the case that  $2l = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$ , the image of  $H_{\mathbb{C}}$  in  $GL_{X_{\sigma_i}}$  may be  $\bigwedge^{2^{k-1}}(SL(2^k))_{\mathbb{C}}$ .

Hence, we either have

$$(1) \quad H_{\mathbb{C}} \subset \prod_{\sigma_i \in \Sigma(L)} SO(X_{\sigma_i})$$

or, in the case that  $2l = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$ ,

$$(2) \quad H_{\mathbb{C}} \subset \prod_{\sigma_i \in \Sigma(L)} SL(2^k)_{\mathbb{C}}.$$



Now, we know that  $H_{\mathbb{C}}$  surjects onto each factor in the above products. It follows that either  $H_{\mathbb{C}}$  is the entire product or  $H_{\mathbb{C}}$  has a factor which is the graph of an isomorphism  $\alpha$  between factors in the above products.

Namely, without loss of generality, if  $H_{\mathbb{C}}$  is not the entire product in (1) or (2), then we may write

$$H_{\mathbb{C}} = \Gamma_{\alpha} \times H',$$

where  $\Gamma_{\alpha}$  is the graph of  $\alpha$  and  $H'$  is some  $\mathbb{C}$ -group.

Now groups of type  $D_m$  for  $m \neq 4$  have outer automorphism group simply  $\mathbb{Z}/2\mathbb{Z}$ . It follows that in the situation of (1), the isomorphism  $\alpha$  is given by  $\alpha(u) = fuf^{-1}$  for some bijective  $\mathbb{Q}$ -homomorphism  $f : X_{\sigma_i} \rightarrow X_{\sigma_j}$ .

Letting  $p_1$  denote the projection map onto the first factor of the graph  $\Gamma_{\alpha}$  and letting  $p_2$  denote the projection map onto the second factor of  $\Gamma_{\alpha}$ , note that for any  $h \in H_{\mathbb{C}}$ , we then have:

$$\begin{aligned} h \cdot f(x) &= p_2(h)(f(x)) \\ &= (\alpha \circ p_1)(h)(f(x)) \\ &= f \circ p_1(h) \circ f^{-1}(f(x)) \\ &= f \circ p_1(h)(x) \\ &= f(h \cdot x) \end{aligned}$$

Hence,  $f$  is an  $H_{\mathbb{C}}$ -module isomorphism between  $X_{\sigma_i}$  and  $X_{\sigma_j}$ .

Similarly, we know groups of type  $A_m$  have only inner automorphisms. Therefore, in the situation of (2), the isomorphism  $\alpha$  is given by  $\alpha(u) = gug^{-1}$  for some bijective  $\mathbb{Q}$ -homomorphism  $g$  between  $\mathbb{C}$ -vector spaces of dimension  $2^k$ . By the same argument as above, the homomorphism  $g$  is an  $H_{\mathbb{C}}$ -module isomorphism. So, this  $H_{\mathbb{C}}$ -module isomorphism  $g$  induces an  $H_{\mathbb{C}}$ -module isomorphism between  $X_{\sigma_i}$  and  $X_{\sigma_j}$ .

Therefore, in both situations (1) and (2), we get an  $H_{\mathbb{C}}$ -module isomorphism between the  $\mathbb{C}$ -vector spaces  $X_{\sigma_i}$  and  $X_{\sigma_j}$ . However, we also know that the endomorphism algebra  $L = [\text{End}_{\mathbb{Q}}(V)]^H$  of  $V$  is commutative. It follows that there can be no such  $H_{\mathbb{C}}$ -module isomorphisms between factors  $X_{\sigma_i}$  and  $X_{\sigma_j}$ . Thus, in both (1) and (2), the group  $H_{\mathbb{C}}$  must be equal to the entire product.

So we have shown that  $H$  is always the Lefschetz group  $R_{L/\mathbb{Q}}SO(LV)$ , except in the case when  $2l = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$ , where  $H$  could either be the Lefschetz group or  $H$  could be the group  $R_{L/\mathbb{Q}}SU(2^k)$ .

Therefore, it remains to prove that when  $2l = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$ , it is indeed possible for both of the above groups to arise as the Hodge group of a Hodge structure of the desired type. The proof of this relies on the following argument based on [9, Corollary 18.3] and the proof of Theorem 3.1 in [27]:

Suppose  $H$  is a  $\mathbb{Q}$ -algebraic group over  $\mathbb{Q}$  and suppose

$$h : \mathbb{U}_1 \rightarrow H$$

is a polarized  $\mathbb{Q}$ -Hodge structure on  $V$ , with polarization  $\langle, \rangle$ , such the Hodge group of  $h$  is contained in  $H$ . Then consider the Mumford-Tate domain  $D_h$  of all  $\mathbb{Q}$ -Hodge structures on  $(V, \langle, \rangle)$  whose Hodge groups are contained in  $H$ . If  $d$  is a generic element in some connected component  $D_h^0$  of  $D_h$ , then the Hodge group  $H_d$  of  $d$  is contained in  $H$  and moreover, this  $H_d$  is completely determined by the data  $(V, \langle, \rangle, D_h^0)$ . Since the action of  $H(\mathbb{Q})$  preserves the data  $(V, \langle, \rangle, D_h^0)$ , the group  $H(\mathbb{Q})$  normalizes  $H_d$ . But because  $H$  is a connected group over the perfect field  $\mathbb{Q}$ , the group  $H(\mathbb{Q})$  is, in fact, Zariski dense in  $H$  [6, Corollary 18.3]. Hence  $H$  normalizes  $H_d$  and thus  $H_d$  is a normal subgroup of  $H$ .

So first let  $H = R_{L/\mathbb{Q}}SO(LV)$ . Let  $r = [L : \mathbb{Q}]$ . Then, when  $2l = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$ , consider the Hodge structure defined by the  $\mathbb{R}$ -homomorphism:

$$h : \mathbb{U}_1 \rightarrow (SO(LV))^r$$

given by:

$$z \in \mathbb{C}^* \mapsto \left( \begin{array}{c|c} z^{-w} \cdot \text{Id}_n & 0 \\ \hline 0 & z^w \cdot \text{Id}_n \end{array} \right)^r.$$

Now by [9, Proposition IV.A.2], the Mumford-Tate domain  $D_h$  is the quotient of  $H(\mathbb{R})$  by the centralizer in  $H(\mathbb{R})$  of the image of  $h$ . Namely  $D_h$  is the homogeneous space

$$(SO(2l)/S(O(l) \times O(l)))^r,$$

where  $S(O(l) \times O(l))$  is the subgroup of  $O(l) \times O(l)$  of determinant 1. Hence, the Mumford-Tate domain  $D_h$  of Hodge structures with Hodge group contained in  $H = R_{L/\mathbb{Q}}SO(LV)$  is nonempty.

Thus a generic Hodge structure  $d$  in  $D_h$  has Hodge group a normal subgroup of  $R_{L/\mathbb{Q}}SO(LV)$ . But the group  $R_{L/\mathbb{Q}}SO(LV)$  is  $\mathbb{Q}$ -simple. By Remark 2.1, the Hodge structure  $d$  cannot have the trivial Hodge group. Therefore  $d$  must have Hodge group  $R_{L/\mathbb{Q}}SO(LV)$ . Thus, the group  $R_{L/\mathbb{Q}}SO(LV)$  indeed arises as the Hodge group of a  $\mathbb{Q}$ -Hodge structure of the desired type. In fact, this is the generic case.

Now it remains to show that when  $2l = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$ , there exists a  $\mathbb{Q}$ -Hodge structure on  $V$  with Hodge numbers  $(n, 0, \dots, 0, n)$  having Hodge group  $R_{L/\mathbb{Q}}SU(2^k)$ . So let  $H = R_{L/\mathbb{Q}}SU(2^k)$  and  $r = [L : \mathbb{Q}]$ . The Hodge structure

$$h : \mathbb{U}_1 \rightarrow (SU(2^k))^r$$

giving the desired decomposition on  $V$  is given by:

$$z \in \mathbb{C}^* \mapsto \left( \begin{array}{c|c} z^{-\frac{w}{2^{k-1}}} \cdot \text{Id}_{2^{k-1}} & 0 \\ \hline 0 & z^{\frac{(2^k-1)w}{2^{k-1}}} \end{array} \right)^r.$$

Then, again by [9, Proposition IV.A.2], the Mumford-Tate domain  $D_h$  of this Hodge structure  $h$  is

$$(SU(2^k)/S(U(2^k-1) \times U(1)))^r,$$

where  $S(U(2^k-1) \times U(1))$  is the subgroup of  $U(2^k-1) \times U(1)$  of elements with determinant 1. So  $D_h$  is nonempty. As before, using Remark 2.1, a generic element of  $D_h$  is a nontrivial normal subgroup of  $H$ , which is a  $\mathbb{Q}$ -simple group. Therefore, indeed, a generic element of  $D_h$  has Hodge group  $R_{L/\mathbb{Q}}SU(2^k)$ .  $\square$

Thus if  $V$  is a simple polarizable  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  and  $L$  is the endomorphism algebra of  $V$ , Proposition 3.2 determines the possible Hodge groups of  $V$  when  $L$  is of Type I and  $\frac{n}{[L:\mathbb{Q}]}$  is odd. We now wish to address some cases when  $L$  is of Type I and  $\frac{n}{[L:\mathbb{Q}]}$  is even. In order to do this we will make repeated use of the following lemma and its corollary.

**Lemma 3.3.** *Let  $G$  be a real semisimple algebraic group. Suppose the homomorphism*

$$h : \mathbb{U}_1 \rightarrow \prod_{i=1}^N (SU(2) \times G)$$

*defines a simple polarizable  $\mathbb{Q}$ -Hodge structure  $V$  of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$  such that the embedding*

$$\prod_{i=1}^N (SU(2) \times G) \hookrightarrow GL(V)$$

is via the products of the standard representations. For  $z \in \mathbb{C}^*$  write:

$$h(z) = \prod_{i=1}^N A_i \times B_i,$$

where  $A_i \in SU(2)$  and  $B_i \in G$ . Then for each  $i$ , either all the eigenvalues of  $A_i$  are equal to 1 or all of the eigenvalues of  $B_i$  are equal to 1.

*Proof.* Let  $m$  be the rank of  $G$ . Assume the lemma holds whenever the rank of  $G$  is less than  $m$ . In particular, suppose for  $m' < m$ , we have a semisimple group  $G'$ . With the notation of the lemma, suppose we have eigenvalues  $\{\lambda'_j\}_{1 \leq j \leq 2}$  of some  $A'_i$  and eigenvalues  $\{\mu'_k\}_{1 \leq k \leq m'}$  of some  $B'_i$ . Then the tensor product of the standard representation of  $SU(2)$  and the standard representation of  $G'$  sends  $A'_i \times B'_i$  to an endomorphism  $\phi'_i$ , having eigenvalues  $\{\lambda'_j \mu'_k\}$ , of a  $2m'$ -dimensional vector space. Then our inductive assumption is that if all of  $\lambda'_j \mu'_k$  are either equal to  $z^w$  or  $z^{-w}$ , then either all of the  $\lambda'_j$  are equal to 1 or all of the  $\mu'_k$  are equal to 1.

Now consider  $A_i \times B_i$ . Let  $\lambda_1, \lambda_2$  be the eigenvalues of  $A_i$  and let  $\mu_1, \dots, \mu_m$  be the eigenvalues of  $B_i$ . Then the tensor product of the standard representation of  $SU(2)$  and the standard representation of  $G$  sends  $A_i \times B_i$  to an endomorphism  $\phi_i$ , having eigenvalues  $\{\lambda_j \mu_k\}$ , of a  $2m$ -dimensional subspace of  $V$ .

Let  $l$  be the number of eigenvalues of  $\phi_i$  that are equal to  $z^w$ , so then  $2m - l$  of the eigenvalues of  $\phi_i$  are equal to  $z^{-w}$ . Observe that by symmetry, we may assume  $0 \leq l \leq m$ .

Note that if  $l = 0$ , then all of the  $\lambda_j \mu_k$  are equal and hence we have  $\lambda_1 = \lambda_2$  and  $\mu_1 = \dots = \mu_m$ . So, in this case, we have  $\lambda_1 = \pm 1$  and  $\mu_1 = \dots = \mu_m$  is an  $m$ th root of unity. However, we know  $h(1) = 1$  and  $h$  varies continuously. Hence, all of the  $\lambda_j$  and all of the  $\mu_k$  are equal to 1. But since we know  $\lambda_j \mu_k = z^{\pm w}$  this is impossible. Therefore, we must have  $l > 0$ .

Now, there are only two possibilities for  $j$  in the  $\lambda_j$  term: either  $j = 1$  or  $j = 2$ . Since  $l$  of the  $\{\lambda_j \mu_k\}$  are equal to  $z^w$  and  $2m - l$  of the  $\{\lambda_j \mu_k\}$  are equal to  $z^{-w}$ , either  $m - \lfloor \frac{l}{2} \rfloor$  of the  $\{\lambda_1 \mu_k\}$  are equal or  $m - \lfloor \frac{l}{2} \rfloor$  of the  $\{\lambda_2 \mu_k\}$  are equal. Without loss of generality, we suppose:

$$\lambda_1 \mu_1 = \dots = \lambda_1 \mu_{m - \lfloor \frac{l}{2} \rfloor}.$$

This implies  $\mu_1 = \dots = \mu_{m - \lfloor \frac{l}{2} \rfloor}$  and hence we also have

$$\lambda_2 \mu_1 = \dots = \lambda_2 \mu_{m - \lfloor \frac{l}{2} \rfloor}.$$

Now if  $\lambda_1 = \lambda_2$ , then since  $A_i \in SU(2)$ , we have  $\lambda_1 = \lambda_2 = \pm 1$ . But, as before, since  $h(1) = 1$  and  $h$  varies continuously, we must have  $\lambda_1 = \lambda_2 = 1$ . Hence the assertion of the lemma holds and so we are done.

Therefore, suppose that  $\lambda_1 \neq \lambda_2$ . So then, the eigenvalues  $\lambda_1 \mu_1 = \dots = \lambda_1 \mu_{m - \lfloor \frac{l}{2} \rfloor}$  are not equal to the eigenvalues  $\lambda_2 \mu_1 = \dots = \lambda_2 \mu_{m - \lfloor \frac{l}{2} \rfloor}$ .

Now suppose  $\lambda_1 \mu_1$  and  $\lambda_2 \mu_1$  are not equal. Consider the remaining  $2\lfloor \frac{l}{2} \rfloor$  eigenvalues  $\lambda_j \mu_k$  for  $1 \leq j \leq 2$  and  $m - \lfloor \frac{l}{2} \rfloor + 1 \leq k \leq 2m$ . We then have that  $l - (m - \lfloor \frac{l}{2} \rfloor)$  of the eigenvalues  $\lambda_j \mu_k$  are equal to  $z^w$  and  $2m - l - (m - \lfloor \frac{l}{2} \rfloor)$  of the eigenvalues  $\lambda_j \mu_k$  are equal to  $z^{-w}$ . But  $\lfloor \frac{l}{2} \rfloor$  is less than  $m$ . So by our induction hypothesis together with our assumption that  $\lambda_1 \neq \lambda_2$ , we have  $\mu_{m - \lfloor \frac{l}{2} \rfloor + 1} = \dots = \mu_m = 1$ . But then, without loss of generality, we have  $\lambda_1 = z^w$  and  $\lambda_2 = z^{-w}$ . It follows that  $\mu_1 = \dots = \mu_{m - \lfloor \frac{l}{2} \rfloor} = 1$ . Hence all of the  $\mu_k$  are equal to 1.

Thus, indeed, either all of the eigenvalues of  $A_i$  or all of the eigenvalues of  $B_i$  are equal to 1, and so, by induction, the lemma is proved.  $\square$

**Corollary 3.4.** *Let  $G$  be a nontrivial semisimple  $\mathbb{Q}$ -group having no semisimple factors isomorphic to  $SU(2)$ . Then, the group:*

$$SU(2) \times G,$$

*with representation the tensor product of the two standard representations, can never be the Hodge group of a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$ .*

*Proof.* Suppose there exists such a Hodge structure  $V$  with Hodge group  $SU(2) \times G$ . Let

$$h : U_1 \rightarrow SU(2) \times G$$

be the corresponding homomorphism. For  $z \in \mathbb{C}^*$  write  $h(z) = A \times B$ , where  $A \in SU(2)$  and  $B \in G$ . Then by Lemma 3.3, either all the eigenvalues of  $A$  are equal to 1 or all of the eigenvalues of  $B$  are equal to 1. By [9, IV.A.2], the Hodge group  $Hg(V)_{\mathbb{R}}$  is contained in the centralizer of the image of  $h$  in  $SU(2) \times G$ . Because  $G$  has no simple factors isomorphic to  $SU(2)$ , the fact that either all the eigenvalues of  $A$  are equal to 1 or all of the eigenvalues of  $B$  are equal to 1 then implies that the Hodge group of  $V$  is actually contained either in  $SU(2)$  or in  $G$ . However, the Hodge group of  $V$  being contained in either  $SU(2)$  or  $G$  contradicts the initial assumption that the Hodge group of  $V$  is equal to  $SU(2) \times G$ .  $\square$

Lemma 3.3 enables us to determine the possible Hodge groups of a simple polarizable  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  and Type I endomorphism algebra  $L$  such that  $\frac{n}{[L:\mathbb{Q}]}$  is equal to 2.

**Proposition 3.5.** *Suppose  $V$  is a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$  having endomorphism algebra  $L$  a totally real number field such that  $\frac{n}{[L:\mathbb{Q}]} = 2$ . Then:*

$$Hg(V) = \begin{cases} R_{L/\mathbb{Q}}Sp(LV) & \text{if } w \text{ is odd} \\ R_{L/\mathbb{Q}}SO(LV) & \text{if } w \text{ is even.} \end{cases}$$

*Proof.* Let  $H = Hg(V)$  be the Hodge group of  $V$ . Since  $L$  is totally real, when  $w$  is even (respectively when  $w$  is odd), the Lefschetz group of  $V$  is

$$Lef(V) = R_{L/\mathbb{Q}}SO(LV)$$

(respectively

$$Lef(V) = R_{L/\mathbb{Q}}Sp(LV)).$$

So we have:

$$H \subseteq R_{L/\mathbb{Q}}SO(LV)$$

(respectively

$$H \subseteq R_{L/\mathbb{Q}}Sp(LV)).$$

Moreover, by Remark 2.2, because  $L$  is of Type I in Albert's classification, the group  $H$  is semisimple. So we may write  $H_{\mathbb{C}}$  as the almost direct product of its simple factors:

$$H_{\mathbb{C}} = H_1 \cdots H_s.$$

Let  $\Sigma(L) = \{\sigma_1, \dots, \sigma_r\}$  be the set of embeddings of  $L$  into  $\mathbb{C}$ . Then the identical representation  $V \otimes_{\mathbb{Q}} \mathbb{C}$  of  $H_{\mathbb{C}}$  decomposes as:

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma_i \in \Sigma(L)} X_{\sigma_i},$$

where the  $\mathbb{C}$ -vector spaces  $X_{\sigma_i}$  are irreducible, orthogonal (respectively symplectic) representations of  $H_{\mathbb{C}}$  of dimension 4. Each  $X_{\sigma_i}$  then decomposes as a tensor product:

$$X_{\sigma_i} = \bigotimes_{j=1}^s X_{\sigma_i, j},$$

where each  $X_{\sigma_i, j}$  is a representation of the factor  $H_j$ . Some of these representations  $X_{\sigma_i, j}$  will be trivial. If the representation  $X_{\sigma_i, j}$  is nontrivial, then it is a self-dual representation, meaning the representation is either symplectic or orthogonal. Additionally, since the total representation  $X_{\sigma_i}$  is orthogonal (respectively symplectic), the number of  $i$  such that  $X_{\sigma_i}$  is symplectic must be even (respectively odd).

As in the proof of Proposition 3.2, by [15, Theorem 3.11] all highest weights occurring in the representations  $X_{\sigma_i, j}$  must be minuscule.

Thus Table 1 yields that the image of  $H_{\mathbb{C}}$  in  $GL_{X_{\sigma_i}}$  is  $SL(2) \times SL(2)$  acting on  $X_{\sigma_i}$  via the standard representation of  $SL(2)$  times itself (respectively  $Sp(4)$  acting on  $X_{\sigma_i}$  via the standard representation).

In the case when  $w$  is even, we have:

$$(3) \quad H_{\mathbb{C}} \subset \prod_{\sigma_i \in \Sigma(L)} SO(4) \cong \prod_{\sigma_i \in \Sigma(L)} (SL(2) \times SL(2)).$$

We know that  $H_{\mathbb{C}}$  surjects onto each factor  $SL(2) \times SL(2)$ . Hence, either  $H_{\mathbb{C}}$  is the entire product or  $H_{\mathbb{C}}$  has a factor which is the graph of an isomorphism  $\alpha$  between factors of the form  $SL(2)$  or between factors of the form  $SL(2) \times SL(2)$ .

Now for an arbitrary simple polarizable even-weight  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  with totally real endomorphism algebra  $L$  satisfying  $\frac{n}{[L:\mathbb{Q}]} = 2$ , consider the corresponding homomorphism

$$(4) \quad h : U_1 \rightarrow \prod_{\sigma_i \in \Sigma(L)} (SU(2) \times SU(2))$$

For  $z \in \mathbb{C}^*$ , write

$$h(z) = \prod_{\sigma_i \in \Sigma(L)} (A_{\sigma_i} \times B_{\sigma_i}),$$

where  $A_{\sigma_i}, B_{\sigma_i} \in SU(2)$ . Then by Lemma 3.3 for each  $\sigma_i$  either  $A_{\sigma_i}$  has eigenvalues  $z^w$  and  $z^{-w}$  and  $B_{\sigma_i}$  has eigenvalue equal to 1 or vice versa. Hence by [9, Proposition IV.A.2] the Mumford-Tate domain  $D_h$  corresponding to this  $h$  is:

$$(SU(2)/U(1))^r$$

As in the proof of Proposition 3.2, the case when  $H_{\mathbb{C}}$  is equal to the Lefschetz group, namely when  $H_{\mathbb{C}}$  is the entire product  $\prod_{\sigma_i \in \Sigma(L)} (SL(2) \times SL(2))$  in (3), is the generic case. Thus there is certainly a Hodge structure with corresponding homomorphism  $h$  and Mumford-Tate domain  $D_h$ .

Now suppose there existed an even-weight  $\mathbb{Q}$ -Hodge structure of the specified type whose Hodge group was strictly contained in the product from (3):

$$\prod_{\sigma_i \in \Sigma(L)} (SL(2) \times SL(2)).$$

Then by the discussion about homomorphisms of the type (4), the corresponding homomorphism  $h'$  of this Hodge structure would still have to be of the form given by

$$h'(z) = \prod_{\sigma_i \in \Sigma(L)} (A'_{\sigma_i} \times B'_{\sigma_i}),$$

where for each  $\sigma_i$  either  $A'_{\sigma_i}$  has eigenvalues  $z^w$  and  $z^{-w}$  and  $B'_{\sigma_i}$  has eigenvalue equal to 1 or vice versa. Namely, the Mumford-Tate domain  $D_{h'}$  would still be

$$(SU(2)/U(1))^r.$$

However, because there exist Hodge structures with the same Hodge numbers whose Hodge group is exactly equal to the entire product

$$\prod_{\sigma_i \in \Sigma(L)} (SL(2) \times SL(2)),$$

we should have  $D_{h'}$  strictly contained in  $D_h$ . So this is a contradiction.

Therefore, the only possible Hodge group for a Hodge structure of this kind is

$$H_{\mathbb{C}} = \prod_{\sigma_i \in \Sigma(L)} (SL(2) \times SL(2)) \cong (R_{L/\mathbb{Q}} SO(LV))_{\mathbb{C}}.$$

Now consider the odd-weight case. In this case we have:

$$(5) \quad H_{\mathbb{C}} \subset \prod_{\sigma_i \in \Sigma(L)} Sp(4)$$

We know that  $H_{\mathbb{C}}$  surjects onto each factor  $Sp(4)$  in the above product. Hence, either  $H_{\mathbb{C}}$  is the entire product or  $H_{\mathbb{C}}$  has a factor which is the graph of an isomorphism  $\alpha$  between copies of  $Sp(4)$ .

Namely, without loss of generality, if  $H_{\mathbb{C}}$  is not the entire product in (5), then we may write

$$H_{\mathbb{C}} = \Gamma_{\alpha} \times H',$$

where  $\Gamma_{\alpha}$  is the graph of  $\alpha$  and  $H'$  is some  $\mathbb{C}$ -group. Let  $p_1$  denote the projection map onto the first factor of  $\Gamma_{\alpha}$  and let  $p_2$  denote the projection map onto the second factor of  $\Gamma_{\alpha}$ .

Now  $Sp(4)$  has only inner automorphisms, hence there exists a bijective  $\mathbb{Q}$ -homomorphism  $f : X_{\sigma} \rightarrow X_{\tau}$  such that  $\alpha$  is given by  $\alpha(u) = fuf^{-1}$ . Moreover, note that for any  $h \in H_{\mathbb{C}}$ , we have

$$\begin{aligned} h \cdot f(x) &= p_2(h)(f(x)) \\ &= (\alpha \circ p_1)(h)(f(x)) \\ &= f \circ p_1(h) \circ f^{-1}(f(x)) \\ &= f \circ p_1(h)(x) \\ &= f(h \cdot x) \end{aligned}$$

Hence  $f$  is an  $H_{\mathbb{C}}$ -module isomorphism. However, there cannot be such an  $H_{\mathbb{C}}$ -module isomorphism because the endomorphism algebra  $L = [\text{End}_{\mathbb{Q}}(V)]^H$  of  $V$  is commutative. Therefore, we must have

$$H_{\mathbb{C}} = \prod_{\sigma_i \in \Sigma(L)} Sp(4) \cong (R_{L/\mathbb{Q}} Sp(LV))_{\mathbb{C}}.$$

□

Therefore, if  $V$  is a simple polarizable  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  and Type I endomorphism algebra  $L$ , Proposition 3.2 determines the possible Hodge groups of  $V$  when  $\frac{n}{[L:\mathbb{Q}]}$  is odd and Proposition 3.5 determines the possible Hodge groups of  $V$  when  $\frac{n}{[L:\mathbb{Q}]}$  is equal to 2. In the two remaining results about the possible Hodge groups of  $V$ , we will make the additional assumption that  $L$  is equal to  $\mathbb{Q}$ , namely that  $[L:\mathbb{Q}]$  is equal to 1. Then, in Proposition 3.6 we determine the possible Hodge groups when  $n$  is twice an odd number and in Proposition 3.7, we determine the possible Hodge groups when  $n$  is equal to 4.

**Proposition 3.6.** *Suppose  $V$  is a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$ , where  $n$  is twice an odd number, such that the endomorphism algebra  $L$  of  $V$  is equal to  $\mathbb{Q}$ . Then:*

$$Hg(V) = \begin{cases} Sp(2n) & \text{if } w \text{ is odd} \\ SO(V) \text{ or } R_{L/\mathbb{Q}} SU(2^k) \text{ (for } k \geq 3 \text{ and } 2n = \binom{2^k}{2^{k-1}}) & \text{if } w \text{ is even} \end{cases}$$

In the case when  $w$  is even, both possible groups occur.

*Proof.* Let  $H = Hg(V)$  be the Hodge group of  $V$ . When  $w$  is even (respectively when  $w$  is odd), the Lefschetz group of  $V$  is

$$Lef(V) = SO(V)$$

(respectively

$$Lef(V) = Sp(V)).$$

So we have:

$$H \subseteq SO(V)$$

(respectively

$$H \subseteq Sp(V)).$$

Moreover, by Remark 2.2, because  $L$  is of Type I in Albert's classification, the group  $H$  is semisimple. So we may write  $H_{\mathbb{C}}$  as the almost direct product of its simple factors:

$$H_{\mathbb{C}} = H_1 \cdots H_s.$$

Now, because the endomorphism algebra of  $V$  is  $\mathbb{Q}$ , the identical representation  $V \otimes_{\mathbb{Q}} \mathbb{C}$  of  $H_{\mathbb{C}}$  is irreducible and decomposes as a tensor product

$$\bigotimes_{j=1}^s X_j,$$

where each  $X_j$  is a representation of the factor  $H_j$ . Some of these representations will be trivial. If the representation  $X_j$  is nontrivial, then the representation is self-dual, meaning it is either a symplectic or an orthogonal representation. Additionally, since the total representation is orthogonal (respectively symplectic), the number of  $j$  such that  $X_j$  is symplectic must be even (respectively odd).

As in the proof of Proposition 3.2, by [15, Theorem 3.11] all highest weights occurring in the representations  $X_j$  must be minuscule. Then, using that  $n$  is twice an odd number, Table 1 yields that the image of  $H_{\mathbb{C}}$  in  $GL(V)$  is one of:

- (1)  $SO(2n)$  acting by the standard representation
- (2)  $SO(2^k)$ , with  $2n = \binom{2^k}{2^{k-1}}$  for  $k \geq 3$ , acting by  $\bigwedge^{2^{k-1}}$  (Standard)
- (3)  $SL(2) \times SO(n)$  acting by the tensor product of the two standard representations

(respectively,

- (1)  $Sp(2n)$  acting by the standard representation
- (2)  $SL(2) \times SO(n)$  acting by the tensor product of the two standard representations
- (3)  $SL(2) \times SL(2^k)$ , with  $n = \binom{2^k}{2^{k-1}}$  for  $k \geq 3$ , acting by the standard representation of  $SL(2)$  tensor  $\bigwedge^{2^{k-1}}$  (Standard)

However, using Corollary 3.4 we may eliminate  $SL(2) \times SO(n)$  (respectively  $SL(2) \times SO(n)$  and  $SL(2) \times SL(2^k)$ ) as possibilities. Hence  $H_{\mathbb{C}}$  must be  $SO(2n)$  or  $SO(2^k)$  (respectively  $Sp(2n)$ ).

As in the proof of proposition 3.2, the case  $H = SO(V)$  is the generic case and thus is always possible. Thus, it remains to show that when  $w$  is even and  $2n = \binom{2^k}{2^{k-1}}$  for  $k \geq 3$ , the Hodge group  $SU(2^k)$  is also possible.

Consider a Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  having corresponding homomorphism

$$h : \mathbb{U}_1 \rightarrow SU(2^k),$$

given by:

$$z \in \mathbb{C}^* \mapsto \left( \begin{array}{c|c} z^{-\frac{w}{2^{k-1}}} \cdot \text{Id}_{2^{k-1}} & 0 \\ \hline 0 & z^{\frac{(2^k-1)w}{2^{k-1}}} \end{array} \right)$$

Then, as in the proof of Proposition 3.2, by [9, Proposition IV.A.2], the Mumford-Tate domain  $D_h$  of this Hodge structure is

$$SU(2^k)/S(U(2^k - 1) \times U(1)),$$

where  $S(U(2^k - 1) \times U(1))$  is the subgroup of  $U(2^k - 1) \times U(1)$  of elements with determinant 1. Hence  $D_h$  is nonempty. Using Remark 2.1, a generic element of  $D_h$  is a nontrivial normal subgroup of  $SU(2^k)$ , which is a  $\mathbb{Q}$ -simple group. Therefore, indeed, a generic element of  $D_h$  has Hodge group  $SL(2^k)$ . Hence,  $SU(2^k)$  can also occur as the Hodge group of a Hodge structure of the desired form.  $\square$

The last case of a simple polarizable  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  and Type I endomorphism algebra that we address is the case when the endomorphism algebra is equal to  $\mathbb{Q}$  and  $n$  is equal to 4.

**Proposition 3.7.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  and Hodge numbers  $(4, 0, \dots, 0, 4)$  having endomorphism algebra  $L$  equal to  $\mathbb{Q}$ . Then:*

$$Hg(V) = \begin{cases} Sp(8) \text{ or } SL(2) \times SO(4) & \text{if } w \text{ is odd} \\ SO(8) \text{ or } SO(7) & \text{if } w \text{ is even} \end{cases}$$

Moreover, all of the above groups occur.

*Proof.* When  $V$  is of odd weight, the result follows using the equivalence of Remark 3.1 together with the analogous statement for abelian fourfolds proved by Moonen and Zarhin [14, 4.1]. Thus we may assume that  $V$  is of even weight.

Let  $H = Hg(V)$  be the Hodge group of  $V$ . When  $w$  is even, the Lefschetz group of  $V$  is

$$Lef(V) = SO(8).$$

So we have:

$$H \subseteq SO(8).$$

Moreover, by Remark 2.2, because  $L$  is of Type I in Albert's classification, the group  $H$  is semisimple. So we may write  $H_{\mathbb{C}}$  as the almost direct product of its simple factors:

$$H_{\mathbb{C}} = H_1 \cdots H_s.$$

Now, because the endomorphism algebra of  $V$  is  $\mathbb{Q}$ , the identical representation  $V \otimes_{\mathbb{Q}} \mathbb{C}$  of  $H_{\mathbb{C}}$  is irreducible and decomposes as a tensor product

$$\bigotimes_{j=1}^s X_j,$$

where each  $X_j$  is a representation of the factor  $H_j$ . Some of these representations will be trivial. If the representation  $X_j$  is nontrivial, then the representation is self-dual, meaning it is either a symplectic or an orthogonal representation. Additionally, since the total representation is orthogonal, the number of  $j$  such that  $X_j$  is symplectic must be even.

As in the proof of Proposition 3.2, by [15, Theorem 3.11] all highest weights occurring in the representations  $X_j$  must be minuscule. Then, Table 1 yields that the image of  $H_{\mathbb{C}}$  in  $GL(V)$  is one of:

- (1)  $SO(8)$  acting by the standard representation,
- (2)  $SO(7)$  acting by the spin representation
- (3)  $SL(2) \times Sp(4)$  acting by the tensor product of the two standard representations

However, by Corollary 3.4 the last case cannot occur as the Hodge group of a Hodge structure of the desired type. Moreover, as in the proof of Proposition 3.2, we know the Hodge group  $SO(8)$  is the generic case and thus can certainly occur as the Hodge group of a Hodge structure of the desired type.



We now show that  $SO(7)$  acting by the spin representation also occurs as the Hodge group of a  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(4, 0, \dots, 0, 4)$ . To construct the spin representation, choose a pair

$$(W, W^*)$$

of maximal isotropic subspaces of  $\mathbb{Q}^7$  equipped with a symmetric bilinear form  $(,)$  such that

$$W \cap W^* = 0.$$

If  $a_1, a_2, a_3$  is a basis for  $W$ , then there is a unique basis  $\alpha_1, \alpha_2, \alpha_3$  of  $W^*$  such that for all  $i$  and  $j$

$$(a_i, \alpha_j) = \delta_{ij}.$$

Now consider the element  $\phi_{x \wedge y}$  of the Lie algebra  $\mathfrak{so}_7$  given by:

$$\phi_{x \wedge y}(v) = 2((y, v)x - (x, v)y).$$

We can identify  $\bigwedge^2 \mathbb{Q}^7$  with the Lie algebra  $\mathfrak{so}_7$  via the map:

$$x \wedge y \mapsto \phi_{x \wedge y}.$$

A Hodge structure

$$h : \mathbb{U}_1 \rightarrow SO(7)$$

giving the desired decomposition on  $V$  via the spin representation is the one induced by the map on Lie algebras sending

$$1 \mapsto 2w(\alpha_1 \wedge a_1).$$

The centralizer in  $SO(7)$  of the image of  $h$  is then  $GL(1)$ . Hence by [9, Proposition IV.A.2] the Mumford-Tate domain  $D_h$  of  $\mathbb{Q}$ -Hodge structures having Hodge group contained in  $SO(7)$  is

$$SO(7)/GL(1).$$

So  $D_h$  is nonempty. Using Remark 2.1, a generic element of  $D_h$  is a nontrivial normal subgroup of  $SO(7)$ , which is a  $\mathbb{Q}$ -simple group. Therefore, indeed, a generic element of  $D_h$  has Hodge group  $SO(7)$ .  $\square$

#### 4. TYPE II/III

We now turn our attention to simple polarizable  $\mathbb{Q}$ -Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$  having endomorphism algebra of Type II or III in Albert's classification of division algebras with positive involution over a number field [3, Chapter X, §11] to achieve analogous results to the Type I case. In particular, if  $L$  is the endomorphism algebra of such a Hodge structure  $V$ , then the cases we addressed in Section 3 were the cases when  $\frac{n}{[L:\mathbb{Q}]}$  was an odd number or equal to 2 and, when  $L$  was equal to  $\mathbb{Q}$ , the cases when  $n$  was twice an odd number or equal to 4.

In the case of  $L$  being of Type II or III, the number analogous to the number  $\frac{n}{[L:\mathbb{Q}]}$  in the Type I case is the number  $\frac{2n}{[L:\mathbb{Q}]}$ . Thus we will similarly determine the Hodge groups when  $\frac{2n}{[L:\mathbb{Q}]}$  is an odd number (Proposition 4.1) or equal to 2 (Proposition 4.2) and, when  $L$  is a quaternion algebra over  $\mathbb{Q}$ , when  $2n$  is twice an odd number (Proposition 4.3).

**Proposition 4.1.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$  having endomorphism algebra  $L$  of  $V$  of Type II or III in Albert's classification and such that  $m = \frac{2n}{[L:\mathbb{Q}]}$  is odd. Let  $B \cong M_m(L^{\text{op}})$  be the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$  and let  $F$  be the subfield of the center of  $L$  fixed by the Rosati involution. Then:*

(1) *If  $L$  is of Type II, then:*

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}} Sp(B, -) & \text{if } w \text{ is odd} \\ R_{F/\mathbb{Q}} O^+(B, -) \text{ or } R_{F/\mathbb{Q}} SU(2^k) \left( \text{with } 2m = \binom{2^k}{2^{k-1}} \text{ for } k \geq 3 \right) & \text{if } w \text{ is even.} \end{cases}$$

(2) If  $L$  is of Type III, then:

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}}O^+(B, -) \text{ or } R_{F/\mathbb{Q}}SU(2^k) & \left( \text{with } 2m = \binom{2^k}{2^{k-1}} \text{ for } k \geq 3 \right) & \text{if } w \text{ is odd} \\ R_{F/\mathbb{Q}}Sp(B, -) & & \text{if } w \text{ is even.} \end{cases}$$

Moreover, all of the above groups occur. Additionally, when  $L$  is of Type II and  $w$  is even or when  $L$  is of Type III and  $w$  is odd, then  $m \geq 3$ .

*Proof.* The restriction on  $m$  when  $L$  is of Type II and  $w$  is even or when  $L$  is of Type III and  $w$  is odd follows from [27, Theorem 3.1].

Let  $H$  be the Hodge group  $Hg(V)$  of  $V$  and write  $[L : \mathbb{Q}] = r$ . The Lefschetz group of  $V$  is

$$Lef(V) = R_{F/\mathbb{Q}}O^+(B, -)$$

(respectively

$$Lef(V) = R_{F/\mathbb{Q}}Sp(B, -))$$

if  $L$  is of Type II and  $w$  is even or if  $L$  is of Type III and  $w$  is odd (respectively if  $L$  is of Type II and  $w$  is odd or if  $L$  is of Type III and  $w$  is even). So we have:

$$H \subseteq R_{F/\mathbb{Q}}O^+(B, -)$$

(respectively

$$H \subseteq R_{F/\mathbb{Q}}Sp(B, -)).$$

Now, the centralizer of  $H_{\mathbb{C}}$  in  $\text{End}_{\mathbb{C}}(V)$  is isomorphic to

$$L \otimes_{\mathbb{Q}} \mathbb{C} \cong (M_2(\mathbb{C}))^{r/4}.$$

Since  $O^+(B, -)$  is an  $F$ -form of  $SO(2m)$  and  $Sp(B, -)$  is an  $F$ -form of  $Sp(2m)$ , we thus have:

$$H_{\mathbb{C}} \subseteq (SO(2m))^{r/4}$$

(respectively

$$H_{\mathbb{C}} \subseteq (Sp(2m))^{r/4}).$$

The representation  $V_{\mathbb{C}}$  of  $(SO(2m))^{r/4}$  (respectively  $(Sp(2m))^{r/4}$ ) is isomorphic to

$$2W_1 \oplus 2W_2 \oplus \cdots \oplus 2W_{r/4},$$

where  $W_i$  is the representation given by composing the  $i$ th projection map on  $(SO(2m))^{r/4}$  (respectively  $(Sp(2m))^{r/4}$ ) with the standard representation of  $SO(2m)$  (respectively  $Sp(2m)$ ).

Since each representation  $W_i$  is nonzero, for each  $i$  we have:

$$[W_i \otimes W_i^*]^{H_{\mathbb{C}}} \neq 0.$$

Hence, in the decomposition of  $[V_{\mathbb{C}} \otimes V_{\mathbb{C}}^*]^{H_{\mathbb{C}}}$ , the term

$$\left[ 4 \bigoplus_{i=1}^{r/4} W_i \otimes W_i^* \right]^{H_{\mathbb{C}}}$$

has dimension greater than or equal to  $r$ . But since the endomorphism algebra of  $V$  is  $L$ , we know the dimension of  $[V_{\mathbb{C}} \otimes V_{\mathbb{C}}^*]^{H_{\mathbb{C}}}$  is exactly equal to  $r$ . Thus, each  $[W_i \otimes W_i^*]^{H_{\mathbb{C}}}$  term has dimension exactly equal to 1. Therefore, the representation of  $H_{\mathbb{C}}$  on  $W_i$  is irreducible.

Now, since  $L$  is of Type II or Type III, we know by Remark 2.2 that  $H$  is semisimple. Let  $H_{\mathbb{C}}^i$  denote the image of  $H_{\mathbb{C}}$  under the  $i$ -th projection map on  $(SO(2m))^{r/4}$  (respectively  $(Sp(2m))^{r/4}$ ). Write  $H_{\mathbb{C}}^i$  as the almost direct product of its simple factors:

$$H_{\mathbb{C}}^i = H_1^i H_2^i \cdots H_{s_i}^i.$$

Let  $W_{i,j}$  denote the corresponding representation of the factor  $H_j^i$ .

Now, some of the representations  $W_{i,j}$  will be trivial, but, since  $W_i$  is an irreducible orthogonal (respectively symplectic) representation, the nontrivial representations  $W_{i,j}$  are self-dual and the number of  $j$  such that  $W_{i,j}$  is symplectic must be even (respectively odd). Furthermore, since  $\dim_{\mathbb{Q}} W_i = 2m$ , where  $m$  is odd, none of the dimensions of the  $W_{i,j}$  can be divisible by 4.

Moreover, using [15, Theorem 3.11], all highest weights of the representations  $W_j^i$  have length less than or equal to 1 and thus are minuscule. We thus make use of Table 1 to obtain the following facts about a simple classical Lie algebra  $\mathfrak{g}$  with minuscule weight  $\lambda$ :

- (1) If  $V(\lambda)$  is self dual, then  $V(\lambda)$  is even-dimensional
- (2) If  $V(\lambda)$  is symplectic with  $\dim(V(\lambda)) \equiv 2 \pmod{4}$ , then  $\mathfrak{g}$  is of type  $C_l$ , where  $l \geq 1$  is odd, and the representation is the standard representation of  $\mathfrak{sp}_{2l}$
- (3) If  $V(\lambda)$  is orthogonal with  $\dim(V(\lambda)) \equiv 2 \pmod{4}$ , then  $\mathfrak{g}$  is of type either:
  - (a)  $D_l$ , where  $l \geq 1$  is odd, and the representation is the standard representation of  $\mathfrak{so}_{2l}$
  - (b)  $A_{2^k-1}$ , where  $k \geq 3$ , and the representation is  $\bigwedge^{2^{k-1}}$  (Standard)

Now since  $\dim_{\mathbb{Q}} W_{i,j} = 2m$  with  $m$  odd, using Fact 1, there can only be one  $j$  such that  $W_{i,j}$  is nontrivial. Moreover, for this  $j$ , the representation  $W_{i,j}$  must be orthogonal (respectively symplectic), since the number of symplectic factors in the decomposition of  $W_i$  had to be even (respectively odd). Thus, applying Fact 3, the image of  $H_{\mathbb{C}}$  in  $GL_{W_i}$  is  $SO(2m)$  or, in the case that  $2m = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$ , the image of  $H_{\mathbb{C}}$  in  $GL_{W_i}$  may be  $\bigwedge^{2^{k-1}}(SL(2^k))$  (respectively, applying Fact 3, the image of  $H_{\mathbb{C}}$  in  $GL(W_i)$  is  $Sp(2m)$ ).

Therefore we have:

$$(6) \quad H_{\mathbb{C}} \subset \prod_{i=1}^{s/4} SO(W_i)$$

or, in the case that  $2m = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$ ,

$$(7) \quad H_{\mathbb{C}} \subset \prod_{i=1}^{s/4} SL(2^k)_{\mathbb{C}}$$

(respectively,

$$(8) \quad H_{\mathbb{C}} \subset \prod_{i=1}^{s/4} Sp(W_i).$$

Moreover, we know that  $H_{\mathbb{C}}$  surjects onto each factor in the products in (6), (7), and (8). Hence, either  $H_{\mathbb{C}}$  is the entire product or  $H_{\mathbb{C}}$  has a factor which is the graph of an isomorphism  $\alpha$  between factors  $SO(W_i)$  and  $SO(W_j)$  or between factors  $SL(2^k)$  and  $SL(2^k)$  (respectively between factors  $Sp(W_i)$  and  $Sp(W_j)$ ).

Namely, without loss of generality, if  $H_{\mathbb{C}}$  is not the entire product in (6), (7), or (8), then we may write:

$$H_{\mathbb{C}} = \Gamma_{\alpha} \times H',$$

where  $\Gamma_{\alpha}$  is the graph of  $\alpha$  and  $H'$  is some  $\mathbb{C}$ -group. Let  $p_1$  denote projection onto the first factor of  $\Gamma_{\alpha}$  and let  $p_2$  denote projection onto the second factor of  $\Gamma_{\alpha}$ .

Now groups of type  $A_l$  and  $C_l$  have only inner automorphisms and groups of type  $D_l$  for  $l \neq 4$  have outer automorphism group simply  $\mathbb{Z}/2\mathbb{Z}$ . Hence, in the case of (6) and (8), the automorphism  $\alpha$  is given by  $\alpha(u) = fuf^{-1}$  for some bijective  $\mathbb{Q}$ -homomorphism  $f : W_i \rightarrow W_j$ . Moreover, note

that for any  $h \in H_{\mathbb{C}}$ , we have:

$$\begin{aligned}
 h \cdot f(x) &= p_2(h)(f(x)) \\
 &= (\alpha \circ p_1)(h)(f(x)) \\
 &= f \circ p_1(h) \circ f^{-1}(f(x)) \\
 &= f \circ p_1(h)(x) \\
 &= f(h \cdot x)
 \end{aligned}$$

Hence,  $f$  is an  $H_{\mathbb{C}}$ -module isomorphism.

Similarly, in the case of (7), the isomorphism  $\alpha$  is given by  $\alpha(u) = gug^{-1}$  for some bijective  $\mathbb{Q}$ -homomorphism  $g$  between  $\mathbb{C}$ -vector spaces of dimension  $2^k$ . By the same argument as above, the homomorphism  $g$  is an  $H_{\mathbb{C}}$ -module isomorphism. Hence, this homomorphism  $g$  induces an  $H_{\mathbb{C}}$ -module isomorphism  $f$  between some  $W_i$  and  $W_j$ .

Therefore, in the situations of (6), (7), and (8), if  $H_{\mathbb{C}}$  is not the entire product, then we get an  $H_{\mathbb{C}}$ -module isomorphism  $f$  between some  $W_i$  and  $W_j$ . However, we also know that the endomorphism algebra  $L \otimes_{\mathbb{Q}} \mathbb{C} = [\text{End}_{\mathbb{C}}(V)]^{H_{\mathbb{C}}}$  of  $V$  splits as:

$$(M_2(\mathbb{C}))^{r/4},$$

and we have  $r/4$  different representations  $W_i$ . Thus, there can be no such  $H_{\mathbb{C}}$ -module isomorphism  $f$ . Therefore, in the situations of (6), (7), and (8), the group  $H_{\mathbb{C}}$  must be the whole product. Namely  $H$  is the Lefschetz group  $R_{F/\mathbb{Q}}O^+(B, -)$  or, in the case that  $2m = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$ , the group  $H$  could also be  $R_{F/\mathbb{Q}}SU(2^k)$  (respectively  $H$  is the Lefschetz group  $R_{F/\mathbb{Q}}Sp(B, -)$ ).

It now remains to show that when  $2m = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$  and either  $L$  is of Type II and  $w$  is even or  $L$  is of Type III and  $w$  is odd, it is indeed possible for both of the groups  $R_{F/\mathbb{Q}}O^+(B, -)$  and  $R_{F/\mathbb{Q}}SU(2^k)$  to arise as the Hodge group of a Hodge structure of the desired type. As in the proof of Proposition 3.2, the case when the Hodge group is equal to the Lefschetz group  $R_{F/\mathbb{Q}}O^+(B, -)$  is the generic case and thus this case is always possible. We thus need to show that the case when the Hodge group is equal to  $R_{F/\mathbb{Q}}SU(2^k)$  can also occur.

The Hodge structure

$$h : \mathbb{U}_1 \rightarrow (SU(2^k))^{r/4}$$

giving the desired decomposition on  $V$  is given by:

$$z \in \mathbb{C}^* \mapsto \left( \begin{array}{c|c} z^{-\frac{w}{2^{k-1}}} \cdot \text{Id}_{2^{k-1}} & 0 \\ \hline 0 & z^{\frac{(2^k-1)w}{2^{k-1}}} \end{array} \right)^{r/4}.$$

Then, by [9, Proposition IV.A.2], the Mumford-Tate domain  $D_h$  of this Hodge structure is:

$$(SU(2^k)/S(U(2^k-1) \times U(1)))^{r/4},$$

where  $S(U(2^k-1) \times U(1))$  is the subgroup of  $U(2^k-1) \times U(1)$  of determinant 1. Hence  $D_h$  is nonempty. Using Remark 2.1, a generic element of  $D_h$  is a nontrivial normal subgroup of  $R_{F/\mathbb{Q}}SU(2^k)$ , which is a  $\mathbb{Q}$ -simple group. Therefore, indeed, a generic element of  $D_h$  has Hodge group equal to  $R_{F/\mathbb{Q}}SU(2^k)$ . So it is indeed possible for  $R_{F/\mathbb{Q}}SU(2^k)$  to occur as the Hodge group of a Hodge structure of the desired type.  $\square$

Thus if  $V$  is a simple polarizable  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  and  $L$  is the endomorphism algebra of  $V$ , Proposition 4.1 determines the possible Hodge groups of  $V$  when  $L$  is of Type II or Type III and  $\frac{2n}{[L:\mathbb{Q}]}$  is odd. We now address some cases when  $L$  is of Type II or Type III and  $\frac{2n}{[L:\mathbb{Q}]}$  is even. We start by addressing the case when  $\frac{2n}{[L:\mathbb{Q}]}$  is equal to 2.

**Proposition 4.2.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$ , having endomorphism algebra  $L$  of  $V$  of Type II or III in Albert's classification and such that  $[L : \mathbb{Q}] = n$ . Let  $B \cong M_2(L^{\text{op}})$  be the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$  and let  $F$  be the subfield of the center of  $L$  fixed by the Rosati involution. Then:*

(1) *If  $L$  is of Type II, then:*

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}}Sp(B, ^-) & \text{if } w \text{ is odd} \\ R_{F/\mathbb{Q}}O^+(B, ^-) & \text{if } w \text{ is even.} \end{cases}$$

(2) *If  $L$  is of Type III, then:*

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}}O^+(B, ^-) & \text{if } w \text{ is odd} \\ R_{F/\mathbb{Q}}Sp(B, ^-) & \text{if } w \text{ is even.} \end{cases}$$

*Proof.* Let  $H$  be the Hodge group  $Hg(V)$  of  $V$ . The Lefschetz group of  $V$  is

$$Lef(V) = R_{F/\mathbb{Q}}O^+(B, ^-)$$

(respectively

$$Lef(V) = R_{F/\mathbb{Q}}Sp(B, ^-))$$

if  $L$  is of Type II and  $w$  is even or if  $L$  is of Type III and  $w$  is odd (respectively if  $L$  is of Type II and  $w$  is odd or if  $L$  is of Type III and  $w$  is even). So we have:

$$H \subseteq R_{F/\mathbb{Q}}O^+(B, ^-)$$

(respectively

$$H \subseteq R_{F/\mathbb{Q}}Sp(B, ^-)).$$

We know that the centralizer of  $H_{\mathbb{C}}$  in  $\text{End}_{\mathbb{C}}(V)$  is isomorphic to

$$L \otimes_{\mathbb{Q}} \mathbb{C} \cong (M_2(\mathbb{C}))^{n/4}.$$

Since  $O^+(B, ^-)$  is an  $F$ -form of  $SO(4)$  and  $Sp(B, ^-)$  is an  $F$ -form of  $Sp(4)$ , we thus have:

$$H_{\mathbb{C}} \subseteq (SO(4))^{n/4}$$

(respectively

$$H_{\mathbb{C}} \subseteq (Sp(4))^{n/4}).$$

The representation  $V_{\mathbb{C}}$  of  $(SO(4))^{n/4}$  (respectively  $(Sp(4))^{n/4}$ ) is isomorphic to

$$2W_1 \oplus 2W_2 \oplus \dots \oplus 2W_{n/4},$$

where  $W_i$  is the representation given by composing the  $i$ th projection map on  $(SO(4))^{n/4}$  (respectively  $(Sp(4))^{n/4}$ ) with the standard representation of  $SO(4)$  (respectively  $Sp(4)$ ).

Since each representation  $W_i$  is nonzero, for each  $i$  we have:

$$[W_i \otimes W_i^*]^{H_{\mathbb{C}}} \neq 0.$$

Hence, in the decomposition of  $[V_{\mathbb{C}} \otimes V_{\mathbb{C}}^*]^{H_{\mathbb{C}}}$ , the term

$$\left[ 4 \bigoplus_{i=1}^{n/4} W_i \otimes W_i^* \right]^{H_{\mathbb{C}}}$$

has dimension greater than or equal to  $n$ . But since the endomorphism algebra of  $V$  is  $L$ , we know the dimension of  $[V_{\mathbb{C}} \otimes V_{\mathbb{C}}^*]^{H_{\mathbb{C}}}$  is exactly equal to  $n$ . Thus, each  $[W_i \otimes W_i^*]^{H_{\mathbb{C}}}$  term has dimension exactly equal to 1. Therefore, the representation of  $H_{\mathbb{C}}$  on  $W_i$  is irreducible.

Now, since  $L$  is of Type II or Type III, we know by Remark 2.2 that  $H$  is semisimple. Let  $H_{\mathbb{C}}^i$  denote the image of  $H_{\mathbb{C}}$  under the  $i$ -th projection map on  $(SO(4))^{n/4}$  (respectively  $(Sp(4))^{n/4}$ ). Write  $H_{\mathbb{C}}^i$  as the almost direct product of its simple factors:

$$H_{\mathbb{C}}^i = H_1^i H_2^i \cdots H_{s_i}^i.$$

Let  $W_{i,j}$  denote the corresponding representation of the factor  $H_j^i$ .

Now, some of the representations  $W_{i,j}$  will be trivial, but, since  $W_i$  is an irreducible orthogonal (respectively symplectic) representation, the nontrivial representations  $W_{i,j}$  are self-dual and the number of  $j$  such that  $W_{i,j}$  is symplectic must be even (respectively odd).

Using [15, Theorem 3.11], all highest weights of the representations  $W_{i,j}$  have length less than or equal to 1 and thus are minuscule. We thus make use of Table 1 to get:

$$(9) \quad H_{\mathbb{C}} \subset \prod_{i=1}^{n/4} SO(4) \cong \prod_{i=1}^{n/4} (SL(2) \times SL(2))$$

(respectively,

$$(10) \quad H_{\mathbb{C}} \subset \prod_{i=1}^{n/4} Sp(4).$$

Moreover, we know that  $H_{\mathbb{C}}$  surjects onto each factor  $SL(2) \times SL(2)$  (respectively,  $H_{\mathbb{C}}$  surjects onto each factor  $Sp(4)$ ). Let us first address the case of (9).

In the case of (9), either  $H_{\mathbb{C}}$  is the entire product in (9) or  $H_{\mathbb{C}}$  has a factor which is the graph of an isomorphism  $\alpha$  between factors of the form  $SL(2)$  or between factors of the form  $SL(2) \times SL(2)$ .

Now for any  $\mathbb{Q}$ -Hodge structure of the form  $V$ , consider the corresponding homomorphism

$$(11) \quad h : U_1 \rightarrow \prod_{i=1}^{n/4} (SU(2) \times SU(2)).$$

For  $z \in \mathbb{C}^*$ , write  $h(z) = \prod_{i=1}^{n/4} (A_i \times B_i)$ , where  $A_i, B_i \in SU(2)$ . Then for each  $i$ , by Lemma 3.3 either  $A_i$  has eigenvalues  $z^w$  and  $z^{-w}$  and  $B_i$  has eigenvalue equal to 1 or vice versa. Hence, by [9, Proposition IV.A.2] the Mumford-Tate domain  $D_h$  of such a Hodge structure  $h$  is

$$(SU(2)/U(1))^{n/4}.$$

The generic case is the case when  $H_{\mathbb{C}}$  is equal to the entire product from (9):

$$\prod_{i=1}^{n/4} (SL(2) \times SL(2)).$$

Thus there is certainly a Hodge structure with corresponding homomorphism  $h$  and Mumford-Tate domain  $D_h$  having Hodge group equal to  $\prod_{i=1}^{n/4} (SL(2) \times SL(2))$ . We have established that such a Hodge structure has Mumford-Tate domain

$$D_h = (SU(2)/U(1))^{n/4}.$$

Now suppose there existed a  $\mathbb{Q}$ -Hodge structure of the form  $V$  such that the Hodge group of this Hodge structure was strictly contained in  $\prod_{i=1}^{n/4} (SL(2) \times SL(2))$ . Then the corresponding homomorphism  $h'$  of this Hodge structure would still have to be of the form of the homomorphism in (11). Namely the Mumford-Tate domain  $D_{h'}$  would be

$$D_{h'} = (SU(2)/U(1))^{n/4}.$$

However, because there exist Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$  whose Hodge group is exactly equal to  $\prod_{i=1}^{n/4} (SL(2) \times SL(2))$ , we should have:

$$D_{h'} \subsetneq D_h.$$

Since we have shown that  $D_{h'} = D_h = (SU(2)/U(1))^{n/4}$ , we have a contradiction.

Therefore, in the case of (9), the only possible Hodge group for a Hodge structure of this kind is the entire product:

$$H_{\mathbb{C}} = \prod_{i=1}^{n/4} (SL(2) \times SL(2)) \cong (R_{F/\mathbb{Q}} O^+(B, -))_{\mathbb{C}}.$$

Namely,  $H$  is the Lefschetz group  $R_{F/\mathbb{Q}} O^+(B, -)$ .

Now consider the case of (10). Since  $H_{\mathbb{C}}$  surjects onto each factor  $Sp(4)$  in the above product, either  $H_{\mathbb{C}}$  is the entire product or  $H_{\mathbb{C}}$  has a factor which is the graph of an isomorphism  $\alpha$  between factors of the form  $Sp(4)$ . Namely, without loss of generality, if  $H_{\mathbb{C}}$  is not the entire product, then we may write:

$$H_{\mathbb{C}} = \Gamma_{\alpha} \times H',$$

where  $\Gamma_{\alpha}$  is the graph of  $\alpha$  and  $H'$  is a  $\mathbb{C}$ -group. Let  $p_1$  denote projection onto the first factor of  $\Gamma_{\alpha}$  and let  $p_2$  denote projection onto the second factor of  $\Gamma_{\alpha}$ .

Now  $Sp(4)$  has only inner automorphisms. Hence  $\alpha$  is given by  $\alpha(u) = fuf^{-1}$  for some bijective  $\mathbb{Q}$ -homomorphism  $f : W_i \rightarrow W_j$ . Moreover, note that for any  $h \in H_{\mathbb{C}}$ , we have:

$$\begin{aligned} h \cdot f(x) &= p_2(h)(f(x)) \\ &= (\alpha \circ p_1)(h)(f(x)) \\ &= f \circ p_1(h) \circ f^{-1}(f(x)) \\ &= f \circ p_1(h)(x) \\ &= f(h \cdot x) \end{aligned}$$

Hence,  $f$  is an  $H_{\mathbb{C}}$ -module isomorphism. However, the endomorphism algebra  $L \otimes_{\mathbb{Q}} \mathbb{C} = [\text{End}_{\mathbb{C}}(V)]_{\mathbb{C}}^H$  of  $V$  splits as:

$$(M_2(\mathbb{C}))^{n/4},$$

and we have  $n/4$  different representations  $W_i$ . Thus, there cannot be such an  $H_{\mathbb{C}}$ -module isomorphism  $f$  between  $W_i$  and  $W_j$ . Therefore, in the case of (10), we must have that  $H_{\mathbb{C}}$  is equal to the entire product. Namely, we must have

$$H_{\mathbb{C}} = \prod_{i=1}^{n/4} Sp(4) \cong (R_{F/\mathbb{Q}} Sp(B, -))_{\mathbb{C}}.$$

So  $H$  is the Lefschetz group  $R_{F/\mathbb{Q}} Sp(B, -)$ . □

Therefore, if  $V$  is a simple polarizable  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  and Type II or Type III endomorphism algebra  $L$ , Proposition 4.1 determines the possible Hodge groups of  $V$  when  $\frac{2n}{[L:\mathbb{Q}]}$  is odd and Proposition 4.2 determines the possible Hodge groups of  $V$  when  $\frac{2n}{[L:\mathbb{Q}]}$  is equal to 2. In order to address an additional case of  $\frac{2n}{[L:\mathbb{Q}]}$  being even, we now make the additional assumption that  $L$  is a quaternion algebra over  $\mathbb{Q}$ . In this case, we determine the possible Hodge groups when  $2n$  is twice an odd number, meaning when  $n$  is four times an odd number.

**Proposition 4.3.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$ , where  $n$  is four times an odd number, such that the endomorphism algebra  $L$  of  $V$  is a quaternion algebra over  $\mathbb{Q}$ . Let  $B \cong M_{n/2}(L^{\text{op}})$  be the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$ . Then:*

(1) If  $L$  is of Type II, then:

$$Hg(V) = \begin{cases} Sp(B, -) & \text{if } w \text{ is odd} \\ O^+(B, -) \text{ or } SU(2^k) & \left( \text{for } k \geq 3 \text{ and } n = \binom{2^k}{2^{k-1}} \right) \text{ if } w \text{ is even.} \end{cases}$$

(2) If  $L$  is of Type III, then:

$$Hg(V) = \begin{cases} O^+(B, -) \text{ or } SU(2^k) & \left( \text{for } k \geq 3 \text{ and } n = \binom{2^k}{2^{k-1}} \right) \text{ if } w \text{ is odd} \\ Sp(B, -) & \text{if } w \text{ is even.} \end{cases}$$

*Proof.* Let  $H$  be the Hodge group  $Hg(V)$  of  $V$ . The Lefschetz group of  $V$  is

$$Lef(V) = O^+(B, -)$$

(respectively

$$Lef(V) = Sp(B, -))$$

if  $L$  is of Type II and  $w$  is even or if  $L$  is of Type III and  $w$  is odd (respectively if  $L$  is of Type II and  $w$  is odd or if  $L$  is of Type III and  $w$  is even). So we have:

$$H \subseteq O^+(B, -)$$

(respectively

$$H \subseteq Sp(B, -)).$$

We know that the centralizer of  $H_{\mathbb{C}}$  in  $\text{End}_{\mathbb{C}}(V)$  is isomorphic to

$$L \otimes_{\mathbb{Q}} \mathbb{C} \cong M_2(\mathbb{C}).$$

Since  $O^+(B, -)$  is a  $\mathbb{Q}$ -form of  $SO(n)$  and  $Sp(B, -)$  is a  $\mathbb{Q}$ -form of  $Sp(n)$ , we thus have:

$$H_{\mathbb{C}} \subseteq SO(n)$$

(respectively

$$H_{\mathbb{C}} \subseteq Sp(n).)$$

The representation  $V_{\mathbb{C}}$  of  $SO(n)$  (respectively  $Sp(n)$ ) is isomorphic to

$$W \oplus W,$$

where  $W$  is the standard representation of  $SO(n)$  (respectively  $Sp(n)$ ).

Since each representation  $W$  is nonzero, we have:

$$[W \otimes W^*]^{H_{\mathbb{C}}} \neq 0.$$

Hence, in the decomposition of  $[V_{\mathbb{C}} \otimes V_{\mathbb{C}}^*]^{H_{\mathbb{C}}}$ , the term

$$[4(W \otimes W^*)]^{H_{\mathbb{C}}}$$

has dimension greater than or equal to 4. But since the endomorphism algebra of  $V$  is  $L$ , we know the dimension of  $[V_{\mathbb{C}} \otimes V_{\mathbb{C}}^*]^{H_{\mathbb{C}}}$  is exactly equal to 4. Thus, the vector space  $[W \otimes W^*]^{H_{\mathbb{C}}}$  has dimension exactly equal to 1. Therefore, the representation of  $H_{\mathbb{C}}$  on  $W$  is irreducible.

Now since  $L$  is of Type II or Type III, we know by Remark 2.2 that  $H$  is semisimple. So we may write  $H_{\mathbb{C}}$  as the almost direct product of its simple factors:

$$H_{\mathbb{C}} = H_1 \cdots H_s.$$

Let  $W_j$  denote the corresponding representation of the factor  $H_j$ .

Now, some of the representations  $W_j$  will be trivial, but, since  $W$  is an irreducible orthogonal (respectively symplectic) representation, the nontrivial representations  $W_j$  are self-dual and the number of  $j$  such that  $W_j$  is symplectic must be even (respectively odd).

Using [15, Theorem 3.11], all highest weights of the representations  $W_j^i$  have length less than or equal to 1 and thus are minuscule. Using that  $n$  is four times an odd number, Table 1 yields that the image of  $H_{\mathbb{C}}$  in  $GL(V)$  is one of:



- (1)  $SO(n)$  acting by the standard representation
- (2)  $SL(2^k)$ , with  $n = \binom{2^k}{2^{k-1}}$  for  $k \geq 3$ , acting by the representation  $\bigwedge^{2^{k-1}}$  (Standard)
- (3)  $SL(2) \times SO(n/2)$  acting by the tensor product of the two standard representations

(respectively,

- (1)  $Sp(n)$  acting by the standard representation
- (2)  $SL(2) \times SO(n/2)$  acting by the tensor product of the two standard representations
- (3)  $SL(2) \times SL(2^k)$ , with  $n/2 = \binom{2^k}{2^{k-1}}$  for  $k \geq 3$ , acting by the standard representation of  $SL(2)$  tensor  $\bigwedge^{2^{k-1}}$  (Standard) )

However, using Corollary 3.4 we may eliminate the group  $SL(2) \times SO(n/2)$  (respectively the groups  $SL(2) \times SO(n/2)$  and  $SL(2) \times SL(2^k)$ ) as possibilities. Hence  $H_{\mathbb{C}}$  must be the groups  $SO(n)$  or  $SL(2^k)$  (respectively the group  $Sp(n)$ ).

It now remains to show that when  $2m = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$  and either  $L$  is of Type II and  $w$  is even or  $L$  is of Type III and  $w$  is odd, it is indeed possible for both the groups  $SO(n)$  and  $SU(2^k)$  to arise as the Hodge group of a Hodge structure of the desired type.

As in the proof of Proposition 3.2, the case when the Hodge group is equal to  $O^+(B, -)$  is the generic case and thus this case is always possible. We thus need to show that the case when the Hodge group is equal to  $SU(2^k)$  can also occur.

The desired such Hodge structure

$$h : \mathbb{U}_1 \rightarrow SU(2^k)$$

on  $V$  is given by:

$$z \in \mathbb{C}^* \mapsto \left( \begin{array}{c|c} z^{-\frac{w}{2^{k-1}}} \cdot \text{Id}_{2^{k-1}} & 0 \\ \hline 0 & z^{\frac{(2^k-1)w}{2^{k-1}}} \end{array} \right).$$

Then, by [9, Proposition IV.A.2], the Mumford-Tate domain  $D_h$  of this Hodge structure is

$$SU(2^k)/S(U(2^k-1) \times U(1)),$$

where  $S(U(2^k-1) \times U(1))$  is the subgroup of  $U(2^k-1) \times U(1)$  of elements with determinant 1. Hence  $D_h$  is nonempty. Using Remark 2.1, a generic element of  $D_h$  is a nontrivial normal subgroup of  $SU(2^k)$ , which is a  $\mathbb{Q}$ -simple group. Therefore, indeed, a generic element of  $D_h$  has Hodge group equal to  $SU(2^k)$ . Hence it is possible for  $SU(2^k)$  to occur as the Hodge group of a Hodge structure of the desired type.  $\square$

## 5. TYPE IV

We now focus on simple polarizable  $\mathbb{Q}$ -Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$  that have endomorphism algebra of Type IV. In general, it is difficult to determine the possible Hodge groups of such Hodge structures. We are able to do this under certain conditions when the endomorphism algebra  $L$  is an imaginary quadratic field as follows.

**Proposition 5.1.** *Suppose  $V$  is a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$  having endomorphism algebra  $L$  an imaginary quadratic field. Moreover, if  $\Sigma(L) = \{\sigma, \bar{\sigma}\}$  is the set of embeddings of  $L$  into  $\mathbb{C}$ , then suppose that the numbers*

$$n_{\sigma} = \dim V^{w,0}(\sigma) \text{ and } n_{\bar{\sigma}} = \dim V^{w,0}(\bar{\sigma})$$

*are coprime. Then, if  $B \cong M_n(L^{\text{op}})$  is the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$ , we have:*

$$Hg(V) = U(B, -).$$

*Proof.* The above result and its proof are analogous to those given in [21, Theorem 3] for simple abelian varieties.

Let  $H = Hg(V)$  be the Hodge group of  $V$ . The Lefschetz group of  $V$  is  $U(B, -)$ , so we have:

$$(12) \quad H \subseteq U(B, -).$$

Thus we need to show the containment in (12) is actually an equality.

Decompose the vector space  $V \otimes_{\mathbb{Q}} \mathbb{C}$  as:

$$(13) \quad V \otimes_{\mathbb{Q}} \mathbb{C} = V(\sigma) \oplus V(\bar{\sigma}),$$

where  $V(\sigma)$  is the  $n$ -dimensional subspace on which  $L$  acts via  $\sigma$  and  $V(\bar{\sigma})$  is the  $n$ -dimensional subspace on which  $L$  acts via  $\bar{\sigma}$ .

Let  $M$  denote the Mumford-Tate group  $MT(V)$  of  $V$ . Consider the map:

$$\rho : M_{\mathbb{C}} \rightarrow \mathrm{GL}_{V(\sigma)}$$

induced by the action of  $M$  on  $V(\sigma)$ . Let  $N$  be the image of  $\rho$ .

Note first of all that  $N$  is a reductive, connected subgroup of  $\mathrm{GL}_{V(\sigma)}$ . Moreover, because  $L = \mathrm{End}_M(V)$ , where  $L \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ , and because the action of  $M$  is compatible with the decomposition in (13), we have:

$$\mathrm{End}_N V(\sigma) = \mathbb{C}.$$

Finally, note that for  $z \in \mathbb{C}^*$  the composition

$$\rho \circ h \circ \mu : \mathbb{G}_m \rightarrow \mathrm{GL}_{V(\sigma)}$$

acts as multiplication by  $z^{-w}$  on  $V^{w,0}(\sigma)$  and as the identity on  $V^{0,w}(\sigma)$ . Namely,  $N$  contains the group of automorphisms of  $V(\sigma)$  that are a homothety on  $V^{w,0}(\sigma)$  and the identity on  $V^{0,w}(\sigma)$ . Finally, since  $n_{\sigma}$  and  $n_{\bar{\sigma}}$  are coprime, the dimensions of  $V^{w,0}(\sigma)$  and  $V^{0,w}(\sigma)$  are coprime.

In summary, we have established the following:

- (1) The group  $N$  is a reductive, connected subgroup of  $\mathrm{GL}_{V(\sigma)}$
- (2)  $\mathrm{End}_N V(\sigma) = \mathbb{C}$
- (3) The group  $N$  contains the group of automorphisms of  $V(\sigma)$  that are a homothety on  $V^{w,0}(\sigma)$  and the identity on  $V^{0,w}(\sigma)$
- (4) The dimensions of  $V^{w,0}(\sigma)$  and  $V^{0,w}(\sigma)$  are coprime

The above observations are exactly the situation of a result of Serre [24, Proposition 5] which establishes that, under these circumstances, we have:

$$N = \mathrm{GL}_{V(\sigma)}.$$

In particular, the fact that  $\rho$  surjects onto  $\mathrm{GL}_{V(\sigma)}$  implies that the commutator subgroup of  $M_{\mathbb{C}}$  surjects onto  $\mathrm{SL}_{V(\sigma)}$ .

The Lefschetz group  $U(B, -)$  of  $V$  is a  $\mathbb{Q}$ -form of  $\mathrm{GL}(n)$ . Hence, by dimension arguments, in order to show that in (12) the group  $H$  is equal to  $U(B, -)$ , it is enough to show that the center of  $M$  has dimension at least 2. To do this, it is enough to produce a two-dimensional torus which is a quotient of  $M$ . Consider the inclusion:

$$(14) \quad M \rightarrow \mathrm{GL}(L V),$$

where  $L V$  denotes  $V$  considered as an  $L$ -vector space. The inclusion in (14) yields a determinant map

$$\delta : M \rightarrow \mathrm{Res}_{L/\mathbb{Q}}(\mathbb{G}_m).$$

Now, the character group of  $\mathrm{Res}_{L/\mathbb{Q}}(\mathbb{G}_m)$  is just the free abelian group on  $\sigma$  and  $\bar{\sigma}$ . Thus

$$\mathrm{Res}_{L/\mathbb{Q}}(\mathbb{G}_m)_{\mathbb{C}} \cong \mathbb{G}_m \times \mathbb{G}_m.$$

The Mumford-Tate group  $M$  contains the torus of homotheties of  $\mathrm{GL}_V$ . In the image over  $\mathbb{C}$  of the map  $\delta$ , this torus of homotheties corresponds to the diagonal  $\mathbb{G}_m$  in the expression  $\mathbb{G}_m \times \mathbb{G}_m$ . The composition

$$\delta \circ \mu : \mathbb{G}_m \rightarrow \mathbb{G}_m \times \mathbb{G}_m$$

is given by:

$$z \in \mathbb{C}^* \mapsto (z^{n_\sigma}, z^{n_{\bar{\sigma}}}).$$

But since  $n_\sigma$  and  $n_{\bar{\sigma}}$  are coprime, clearly  $n_\sigma$  and  $n_{\bar{\sigma}}$  are not equal. Thus the image of the composition  $\delta \circ \mu$  is a subtorus of  $\mathbb{G}_m \times \mathbb{G}_m$  which is not the diagonal. Hence  $\delta$  is surjective. So, the Mumford-Tate group  $M$  is indeed the quotient of a two dimensional torus and thus  $H$  is equal to the Lefschetz group  $U(B, -)$ .  $\square$

**5.1.  $E$ -Hodge Structures.** Proposition 5.1 determines the possible Hodge group of a simple polarizable  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  with endomorphism algebra an imaginary quadratic field under certain conditions. In general however, it is difficult to say much about the possible Hodge groups of a simple polarizable  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  with arbitrary Type IV endomorphism algebra.

A requirement on the endomorphism algebra  $L$  of a general simple polarizable  $\mathbb{Q}$ -Hodge structure  $V$  with Hodge numbers  $(n, 0, \dots, 0, n)$  that makes it easier to determine the possible Hodge structures is the requirement that  $V$  also be an  $E$ -Hodge structure with Hodge numbers  $(m, 0, \dots, 0, m)$  for  $E$  a CM field. The reason this additional requirement makes determining the possible Hodge groups easier stems from the following lemma.

**Lemma 5.2.** *Let  $V$  be a polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$  and endomorphism algebra  $L$  such that there exists a CM field  $E$  with an embedding*

$$E \rightarrow L$$

*If  $[E : \mathbb{Q}] = r$  then write  $2n = lr$ . Let  $J$  be the maximal totally real subfield of  $E$ , and let  $C \cong M_l(E^{\mathrm{op}})$  be the centralizer of  $E$  in  $\mathrm{End}_{\mathbb{Q}}(V)$ . Then we have:*

$$Hg(V) \subseteq R_{J/\mathbb{Q}}SU(C, -)$$

*if and only if  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(\frac{n}{r}, 0, \dots, 0, \frac{n}{r})$ .*

*Proof.* We know  $V \otimes_{\mathbb{Q}} \mathbb{C}$  has rank  $l$  as a free  $E \otimes_{\mathbb{Q}} \mathbb{C}$ -module, so consider the exterior product

$$\bigwedge_E^l V.$$

Then an element  $h$  in  $Hg(V)$  acts on  $\bigwedge_E^l V$  as multiplication by  $\mathrm{Norm}_E(h)$ . Since  $E$  is contained in  $L$ , we already know that  $Hg(V)$  is contained in  $R_{J/\mathbb{Q}}U(C, -)$ . Hence we have

$$Hg(V) \subseteq R_{J/\mathbb{Q}}SU(C, -)$$

if and only if  $\bigwedge_E^l V$  is invariant under the action of  $Hg(V)$ , meaning if and only if  $\bigwedge_E^l V$  is purely of type  $(\frac{l}{2}, \frac{l}{2})$  as a Hodge substructure of  $\bigwedge_{\mathbb{Q}}^l V$ .

Now consider the decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in \Sigma(E)} (V^{w,0}(\sigma) \oplus V^{0,w}(\sigma)).$$

In this decomposition, for each  $\sigma \in \Sigma(E)$ , let

$$n_\sigma = \dim V^{w,0}(\sigma).$$

Then we have,

$$\begin{aligned}
\left( \bigwedge_E^l V \right) \otimes_{\mathbb{Q}} \mathbb{C} &= \bigwedge_{E \otimes_{\mathbb{Q}} \mathbb{C}}^l (V \otimes_{\mathbb{Q}} \mathbb{C}) \\
&= \bigoplus_{\sigma \in \Sigma(E)} \bigwedge^l (V^{w,0}(\sigma) \oplus V^{0,w}(\sigma)) \\
&= \bigoplus_{\sigma \in \Sigma(E)} \bigoplus_{n_{\sigma}} \left( \bigwedge^{n_{\sigma}} V^{w,0}(\sigma) \otimes \bigwedge^{l-n_{\sigma}} V^{0,w}(\sigma) \right)
\end{aligned}$$

Thus  $\bigwedge_E^l V$  is purely of type  $(\frac{l}{2}, \frac{l}{2})$  if and only if  $n_{\sigma} = \frac{l}{2} = l - n_{\sigma}$  for all embeddings  $\sigma \in \Sigma(E)$  of  $E$  into  $\mathbb{C}$ . But the property of having  $n_{\sigma} = \frac{l}{2} = l - n_{\sigma}$  for all embeddings  $\sigma \in \Sigma(E)$  is exactly what it means for  $V$  to be an  $E$ -Hodge structure with Hodge numbers  $(\frac{n}{r}, 0, \dots, 0, \frac{n}{r})$ , so this finishes the proof.  $\square$

Lemma 5.2 establishes a fundamental restriction on the Hodge group of a simple polarizable  $\mathbb{Q}$ -Hodge structure  $V$  with Hodge numbers  $(n, 0, \dots, 0, n)$  having the property that  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(m, 0, \dots, 0, m)$  for some  $m$ . This restriction on the Hodge group allows us to say something about the possible Hodge groups of simple polarizable  $\mathbb{Q}$ -Hodge structures that are also  $E$ -Hodge structures. In particular, if  $V$  is such a Hodge structure and  $L$ , which is of Type IV in Albert's classification, is the endomorphism algebra of  $V$ , we may consider the possible Hodge groups under various conditions on  $[L : E]$ . The first case we consider is the case when  $[L : E]$  is equal to 1, namely when the endomorphism algebra of  $V$  is equal to  $E$ . For this case, we make the additional assumption that  $n$  is prime.

**Proposition 5.3.** *Let  $V$  be a simple polarizable  $E$ -Hodge structure with Hodge numbers  $(p, 0, \dots, 0, p)$ , where  $p$  is prime and  $E$  is a CM field of degree  $r$  over  $\mathbb{Q}$ . Suppose the endomorphism algebra of  $V$  as a  $\mathbb{Q}$ -Hodge structure is  $E$ . Let  $C \cong M_{2p}(E^{\text{op}})$  be the centralizer of  $E$  in  $\text{End}_{\mathbb{Q}}(V)$  and let  $J$  be the maximal totally real subfield of  $E$ . Then:*

$$Hg(V) = R_{J/\mathbb{Q}} SU(C, ^-).$$

*Proof.* Let  $H$  be the Hodge group  $Hg(V)$  of  $V$ . Since  $V$  is an  $E$ -Hodge structure, by Lemma 5.2 we know:

$$H \subseteq R_{J/\mathbb{Q}} SU(C, ^-).$$

Since  $SU(C, ^-)$  is a  $J$ -form of  $SL(2p)$  we have:

$$H_{\mathbb{C}} \subseteq (SL(2p))^{r/2}$$

The representation  $V_{\mathbb{C}}$  of  $(SL(2p))^{r/2}$  is isomorphic to

$$(W_1 \oplus W_1)^* \oplus (W_2 \oplus W_2^*) \oplus \dots \oplus (W_{r/2} \oplus W_{r/2}^*)$$

where  $W_i$  is the representation given by composing the  $i$ th projection map on  $(SL(2p))^{r/2}$  with the standard representation of  $SL(2p)$ .

Since each representation  $W_i$  is nonzero, for each  $i$  we have:

$$[W_i \otimes W_i^*]^{H_{\mathbb{C}}} \neq 0.$$

Hence, in the decomposition of  $[V_{\mathbb{C}} \otimes V_{\mathbb{C}}^*]^{H_{\mathbb{C}}}$ , the term

$$\left[ 2 \bigoplus_{i=1}^{r/2} W_i \otimes W_i^* \right]^{H_{\mathbb{C}}}$$

has dimension greater than or equal to  $r$ . But since the endomorphism algebra of  $V$  is  $E$ , we know  $[V_{\mathbb{C}} \otimes V_{\mathbb{C}}^*]^{H_{\mathbb{C}}}$  has dimension equal to  $r$ . Thus, for each  $i$ , the term

$$[W_i \otimes W_i^*]^{H_{\mathbb{C}}}$$

has dimension exactly equal to 1 and we have:

$$[W_i^{\otimes 2}]^{H_{\mathbb{C}}} = 0.$$

Therefore, the representation of  $H_{\mathbb{C}}$  on  $W_i$  is irreducible and non-self-dual.

Moreover, since  $E = [\text{End}_{\mathbb{Q}}(V)]^H$ , we know that the center of  $H$  is contained in the center of  $E$ , which in this case is all of  $E$ . Hence the center of  $H$  is contained in  $R_{J/\mathbb{Q}}SU(B, -) \cap E$ . So elements of the center of  $H$  must be  $2n$ -th roots of unity in  $E$ . However there are only finitely many  $2n$ -th roots of unity in  $E$ . Thus, the center of  $H$  is finite. Combining this with the fact that  $H$  is reductive yields that  $H$  is semisimple.

Let  $H_{\mathbb{C}}^i$  denote the image of  $H_{\mathbb{C}}$  under the  $i$ -th projection map on  $(SL(2p))^{r/2}$ . Write  $H_{\mathbb{C}}^i$  as the almost direct product of its simple factors:

$$H_{\mathbb{C}}^i = H_1^i H_2^i \cdots H_{s_i}^i.$$

Let  $W_{i,j}$  denote the corresponding representation of the factor  $H_j^i$ .

Now, using [15, Theorem 3.11], all highest weights of the representations  $W_{i,j}$  have length less than or equal to 1 and thus are minuscule. Hence we make use of Table 1. We know that the representation  $W_i$  is not self-dual, therefore without loss of generality we may assume that the representation  $W_{i,1}$  is not self-dual. Moreover, the dimension of  $W_{i,1}$  must divide  $2p$ . However, referring to Table 1, the representation  $W_{i,1}$  cannot have dimension 1 or 2.

If  $W_{i,1}$  has dimension  $2p$ , then  $H^i$  is simple. Using Table 1 yields the following options for  $H^i$ :

- (1) The group  $SL(2p)$  acting on  $W_i$  by the standard representation
- (2) When  $p = 5$ , the group  $SL(5)$  acting on  $W_i$  by  $\wedge^2(\text{Standard})$ .

Note that arriving at the above options requires the combinatorial fact that a binomial coefficient  $\binom{n}{k}$  is equal to  $2p$  for  $p$  a prime if and only if  $k = 1$  or  $k = 2$  and  $n = 4$  or  $5$ . At first glance, Table 1 would indicate that the 6-dimensional representation of  $SL(4)$  given by  $\wedge^2(\text{Standard})$  would be an option as well. However, this representation is self-dual, and thus is not a possibility.

If  $W_{i,1}$  has dimension  $p$ , then  $H_1^i$  is  $SL(p)$  acting on  $W_{i,1}$  by the standard representation. Then, without loss of generality,  $H_2^i$  is  $SL(2)$  acting on  $W_{i,2}$  by the standard representation. Note that since  $V_i$  is not self-dual, we must have  $p$  not equal to 2.

Thus we have three possibilities for  $H^i$  acting on  $W_i$ :

- (1) The group  $SL(2p)$  acting by the standard representation
- (2) When  $p = 5$ , the group  $SL(5)$  acting by  $\wedge^2(\text{Standard})$
- (3) When  $p \neq 2$ , the group  $SL(2) \times SL(p)$  acting by the product of the standard representations

In the above cases, we have corresponding inclusions:

- (1)  $H_{\mathbb{C}} \subset \prod_{i=1}^{r/2} SL(2p)$
- (2)  $H_{\mathbb{C}} \subset \prod_{i=1}^{r/2} SL(5)$
- (3)  $H_{\mathbb{C}} \subset \prod_{i=1}^{r/2} (SL(2) \times SL(p))$ .

Moreover, we know that  $H_{\mathbb{C}}$  surjects onto each factor in the above products.

Consider the first two cases first. Without loss of generality, if  $H_{\mathbb{C}}$  is not the entire product in cases (1) and (2), then we may write:

$$H_{\mathbb{C}} = \Gamma_{\alpha} \times H',$$

where  $\Gamma_{\alpha}$  is the graph of  $\alpha$ , an isomorphism between factors  $SL(2p)$  or between factors  $SL(5)$ , and  $H'$  is some  $\mathbb{C}$ -group. Let  $p_1$  denote projection onto the first factor of  $\Gamma_{\alpha}$  and let  $p_2$  denote projection onto the second factor of  $\Gamma_{\alpha}$ .

Now groups of type  $A_l$  have only inner automorphisms. Hence, the automorphism  $\alpha$  is given by  $\alpha(u) = fuf^{-1}$  for some bijective  $\mathbb{Q}$ -homomorphism  $f : W_i \rightarrow W_j$ . Moreover, note that for any  $h \in H_{\mathbb{C}}$ , we have

$$\begin{aligned} h \cdot f(x) &= p_2(h)(f(x)) \\ &= (\alpha \circ p_1)(h)(f(x)) \\ &= f \circ p_1(h) \circ f^{-1}(f(x)) \\ &= f \circ p_1(h)(x) \\ &= f(h \cdot x) \end{aligned}$$

Hence,  $f$  is an  $H_{\mathbb{C}}$ -module isomorphism.

However, we know that the endomorphism algebra  $L = [\text{End}_{\mathbb{Q}}(V)]^H$  of  $V$  is commutative. Thus, there can be no  $H_{\mathbb{C}}$ -module isomorphism between any  $W_i$  and  $W_j$ . Therefore, in the situations of (1) and (2), the group  $H_{\mathbb{C}}$  must be equal to the entire product.

Namely, the Hodge group of  $V$  is  $R_{J/\mathbb{Q}}SU(C, -)$  or, when  $p = 5$ , we also have the possibility that the Hodge group could be  $R_{J/\mathbb{Q}}SU(5)$ . We now use a combinatorial argument to show that this latter possibility cannot actually occur.

Suppose that  $p = 5$  and that there exists an  $E$ -Hodge structure  $V$  with Hodge numbers  $(5, 0, \dots, 0, 5)$  having Hodge group  $R_{J/\mathbb{Q}}SU(5)$ . The group  $R_{J/\mathbb{Q}}SU(5)$  embeds into  $R_{J/\mathbb{Q}}SU(C, -)$  via  $\bigwedge^2$ . Thus let

$$h : U_1 \rightarrow R_{J/\mathbb{Q}}SU(5)$$

be the homomorphism corresponding to  $V$ . For  $z \in \mathbb{C}^*$ , let  $\lambda_1, \dots, \lambda_5$  be the eigenvalues of  $h(z)$ . Then the image of  $h(z)$  in  $R_{J/\mathbb{Q}}SU(C, -)$  has eigenvalues  $\lambda_i \lambda_j$  for  $1 \leq i < j \leq 5$ . Now five of these eigenvalues  $\lambda_i \lambda_j$  are equal to  $z^{-w}$  and five of them are equal to  $z^w$ .

Of these eigenvalues  $\lambda_i \lambda_j$ , let us consider the four:  $\lambda_1 \lambda_2, \lambda_1 \lambda_3, \lambda_1 \lambda_4, \lambda_1 \lambda_5$ . By symmetry, the only distinct cases to consider are the case when all four of these are equal, the case when exactly three of them are equal, and the case when exactly two of them are equal.

Suppose four of the eigenvalues  $\lambda_i \lambda_j$  are equal. Namely we have:

$$\lambda_1 \lambda_2 = \lambda_1 \lambda_3 = \lambda_1 \lambda_4 = \lambda_1 \lambda_5.$$

This implies

$$\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5.$$

Moreover, there exists some  $\lambda_i \lambda_j$  with  $i, j \in \{2, 3, 4, 5\}$  such that

$$\lambda_1 \lambda_2 = \lambda_1 \lambda_3 = \lambda_1 \lambda_4 = \lambda_1 \lambda_5 = \lambda_i \lambda_j.$$

But then we have:

$$\lambda_i \lambda_j = \lambda_2^2 = \lambda_1 \lambda_2.$$

Hence  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5$ . However, this contradicts the fact that five of the eigenvalues  $\lambda_i \lambda_j$  are equal to  $z^{-w}$  and five of them are equal to  $z^w$ . So the case when four of the eigenvalues  $\lambda_i \lambda_j$  are equal is impossible.

Now suppose exactly three of the eigenvalues  $\lambda_i \lambda_j$  are equal. Namely without loss of generality,

$$\lambda_1 \lambda_2 = \lambda_1 \lambda_3 = \lambda_1 \lambda_4 = (\lambda_1 \lambda_5)^{-1}.$$

Then we have:

$$\lambda_2 = \lambda_3 = \lambda_4$$

and

$$(\lambda_1)^2 = \lambda_2^{-1} \lambda_5^{-1}.$$

We know there are two eigenvalues  $\lambda_i \lambda_j$  such that  $\lambda_i \lambda_j = \lambda_1 \lambda_2$  but  $\{(i, j)\} \notin \{(1, 2), (1, 3), (1, 4), (1, 5)\}$ . Hence at least one of these two  $\lambda_i \lambda_j$  has both  $i$  and  $j$  in  $\{2, 3, 4\}$ . Then we get

$$\lambda_2^2 = \lambda_1 \lambda_2,$$

which implies

$$\lambda_1 = \lambda_2,$$

and thus

$$(\lambda_1)^3 = (\lambda_5)^{-1}.$$

But since  $h(z)$  is in  $R_{J/\mathbb{Q}}SU(5)$ , we must have  $\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 = 1$ , and thus  $(\lambda_1)^4 = (\lambda_5)^{-1}$ . So then  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1$ , which is impossible.

Finally, suppose exactly two of the  $\lambda_1 \lambda_j$  are equal. Namely without loss of generality, suppose we have:

$$\lambda_1 \lambda_2 = \lambda_1 \lambda_3 = (\lambda_1 \lambda_4)^{-1} = (\lambda_1 \lambda_5)^{-1}.$$

Then  $\lambda_2 = \lambda_3$  and  $\lambda_4 = \lambda_5$ . But then we have:

$$\lambda_2 \lambda_4 = \lambda_2 \lambda_5 = \lambda_3 \lambda_4 = \lambda_3 \lambda_5,$$

which implies that there are at least six equal  $\lambda_i \lambda_j$ , which is impossible.

Hence we have shown that there can be no homomorphism

$$h : U_1 \rightarrow R_{J/\mathbb{Q}}SU(5)$$

yielding a simple polarizable Hodge structure with Hodge numbers  $(5, 0, \dots, 0, 5)$ . Therefore the group  $R_{J/\mathbb{Q}}SU(5)$  acting on  $V$  by  $\bigwedge^2(\text{Standard})$  cannot occur as the Hodge group of a Hodge structure of the desired type.

We now turn our attention to the possibility in (3):

$$H_{\mathbb{C}} \subset \prod_{i=1}^{r/2} (SL(2) \times SL(p)).$$

This inclusion implies that  $H_{\mathbb{C}}$  is equal to a product of factors of the form  $SL(2) \times SL(p)$ , where some of these factors may consist of graphs of isomorphisms between factors  $SL(2)$ ,  $SL(p)$ , or  $SL(2) \times SL(p)$ .

Suppose there exists a Hodge structure  $V$  of the desired type having complexified Hodge group  $H_{\mathbb{C}}$ . Then  $V$  has a corresponding homomorphism over  $\mathbb{C}$ :

$$h : U_1 \rightarrow \prod_{i=1}^{r/2} (SU(2) \times SU(p)).$$

For any  $z \in \mathbb{C}^*$ , write

$$h(z) = \prod_{i=1}^{r/2} (A_i \times B_i),$$

where  $A_i \in SU(2)$  and  $B_i \in SU(p)$ . Then, by Lemma 3.3, for each  $i$  either all of the eigenvalues of  $A_i$  are equal to 1 or all of the eigenvalues of  $B_i$  are equal to 1. Namely, if  $\lambda_j$  for  $1 \leq j \leq 2$  are the eigenvalues of  $A_i$  and  $\mu_k$  for  $1 \leq k \leq p$  are the eigenvalues of  $B_i$ , then either  $\lambda_1 = \lambda_2 = 1$  or  $\mu_1 = \dots = \mu_p = 1$ .

However, the eigenvalues of  $A_i \times B_i$  are  $\lambda_j \mu_k$ . Because the Hodge structure  $V$  has Hodge numbers of the form  $(n, 0, \dots, 0, n)$ , half of the  $\lambda_j \mu_k$  must be equal to  $z^w$  and half of the  $\lambda_j \mu_k$  must be equal to  $z^{-w}$ .

So if all of the eigenvalues  $\lambda_j$  are equal to 1, then half of the  $\mu_k$  are equal to  $z^w$  and half are equal to  $z^{-w}$ . However, since  $B_i$  has rank  $p$  with  $p$  odd, this is impossible.

Therefore, for each  $i$ , the eigenvalues of  $B_i$  must all be equal to 1. By [9, IV.A.2], the Hodge group  $H_{\mathbb{R}}$  is contained in the centralizer of the image of  $h$  in  $\prod_{i=1}^{r/2} (SU(2) \times SU(p))$ . Since  $p \neq 2$ , the fact that the eigenvalues of  $B_i$  must all be equal to 1 implies that we have  $H_{\mathbb{C}} \subset \prod_{i=1}^{r/2} SL(2)$ . However this contradicts the fact that  $H_{\mathbb{C}}$  must surject onto  $\frac{r}{2}$  copies of  $SL(2) \times SL(p)$ . Thus, no such Hodge structure  $V$  can occur.

Hence the only possibility for the Hodge group of a Hodge structure of the desired type is the group  $R_{J/\mathbb{Q}}SU(C, -)$ .  $\square$

Thus Proposition 5.3 determines the Hodge group of a simple polarizable  $\mathbb{Q}$ -Hodge structure  $V$  which is also an  $E$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  when  $[L : E]$  is equal to 1 and  $p$  is prime. Proposition 5.4 now determines the Hodge group of such a Hodge structure when  $[L : E]$  is equal to 2 under an additional condition on the action of the endomorphism algebra  $L$  on  $V$ .

**Proposition 5.4.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  such that there exists a CM field  $E$  such that  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$ . Write  $[E : \mathbb{Q}] = r$  and suppose that the endomorphism algebra  $L$  of  $V$  is of Type IV and  $[L : \mathbb{Q}] = 2r$ . Moreover for all embeddings  $\sigma \in \Sigma(L)$  of  $L$  into  $\mathbb{C}$  suppose that*

$$n_{\sigma} = \dim V^{w,0}(\sigma) \text{ and } n_{\bar{\sigma}} = \dim V^{w,0}(\bar{\sigma})$$

*are coprime. Then, if  $B \cong M_n(L^{\text{op}})$  is the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$  and  $F$  is the subfield of the center of  $L$  fixed by the Rosati involution, we have:*

$$Hg(V) = R_{F/\mathbb{Q}}SU(B, -).$$

*Proof.* Let  $H = Hg(V)$  be the Hodge group of  $V$ . We know that  $L \otimes_{\mathbb{Q}} \mathbb{C}$  is isomorphic to  $2r$  copies of  $\mathbb{C}$  indexed over the elements of  $\Sigma(L)$ . Thus the action of  $L \otimes_{\mathbb{Q}} \mathbb{C}$  on  $V \otimes_{\mathbb{Q}} \mathbb{C}$  yields a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in \Sigma(L)} V(\sigma),$$

where  $V(\sigma)$  is the  $n$ -dimensional subspace on which  $L$  acts via  $\sigma$ .

Let  $M$  denote the Mumford-Tate group  $MT(V)$  of  $V$ . Consider the map

$$\rho : M_{\mathbb{C}} \rightarrow \text{GL}_{V(\sigma)}$$

induced by the action of  $M$  on  $V(\sigma)$ . Let  $N$  be the image of  $\rho$ .

As in the proof of Proposition 5.1, note first of all that  $N$  is a reductive, connected subgroup of  $\text{GL}_{X_{\sigma}}$ . Moreover, because  $L = \text{End}_M(V)$ , where  $L \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^{2r}$ , and because the action of  $M$  is compatible with the decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in \Sigma(L)} V(\sigma),$$

we have:

$$\text{End}_N V(\sigma) = \mathbb{C}.$$

Finally, note that for  $z \in \mathbb{C}^*$  the composition

$$\rho \circ h \circ \mu : \mathbb{G}_m \rightarrow \text{GL}_{V(\sigma)}$$

acts as multiplication by  $z^{-w}$  on  $V^{w,0}(\sigma)$  and as the identity on  $V^{0,w}(\sigma)$ . Namely,  $N$  contains the group of automorphisms of  $V(\sigma)$  that are a homothety on  $V^{w,0}(\sigma)$  and the identity on  $V^{0,w}(\sigma)$ . Finally, since  $n_{\sigma}$  and  $n_{\bar{\sigma}}$  are coprime, the dimensions of  $V^{w,0}(\sigma)$  and  $V^{0,w}(\sigma)$  are coprime. In summary, we have established the following:

- (1) The group  $N$  is a reductive, connected subgroup of  $\text{GL}_{V(\sigma)}$
- (2)  $\text{End}_N V(\sigma) = \mathbb{C}$



- (3) The group  $N$  contains the group of automorphisms of  $V(\sigma)$  that are a homothety on  $V^{w,0}(\sigma)$  and the identity on  $V^{0,w}(\sigma)$
- (4) The dimensions of  $V^{w,0}(\sigma)$  and  $V^{0,w}(\sigma)$  are coprime

The above observations are exactly the situation of a result of Serre [24, Proposition 5] which establishes that, under these circumstances, we have:

$$N = \mathrm{GL}_{V(\sigma)}.$$

In particular, the fact that  $\rho$  surjects onto  $\mathrm{GL}_{V(\sigma)}$  implies that the commutator subgroup of  $M_{\mathbb{C}}$  surjects onto  $\mathrm{SL}_{V(\sigma)}$ .

Now, the Lefschetz group of  $V$  is  $R_{F/\mathbb{Q}}U(B, -)$ . Hence we have:

$$(15) \quad R_{F/\mathbb{Q}}SU(B, -) \subseteq H \subseteq R_{F/\mathbb{Q}}U(B, -),$$

where  $U(B, -)$  is an  $F$ -form of  $GL(n)$  and  $SU(B, -)$  is an  $F$ -form of  $SL(n)$ .

However,  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$ . Thus by Lemma 5.2 we have:

$$H \subseteq R_{J/\mathbb{Q}}SU(C, -),$$

where  $J$  is the subfield of  $E$  fixed by complex conjugation and  $C \cong M_{2n}(E^{\mathrm{op}})$  is the centralizer of  $E$  in  $\mathrm{End}_{\mathbb{Q}}(V)$ . Here, the group  $SU(C, -)$  is a  $J$ -form of  $SL(2n)$ .

Now since  $L = [\mathrm{End}_{\mathbb{Q}}(V)]^H$ , the center of  $H$  is contained in  $F_0$ , the center of  $L$ . Hence the center of  $H$  is contained in  $R_{F/\mathbb{Q}}U(B, -) \cap F_0$ . So if  $\lambda_1$  and  $\lambda_2$  are elements of the center of  $H$ , both must have norm 1. Moreover, because  $H$  is contained in  $R_{J/\mathbb{Q}}SU(C, -)$ , considering an embedding of  $R_{F/\mathbb{Q}}U(B, -) \cap F_0$  into  $R_{J/\mathbb{Q}}SU(C, -)$ , we must have  $\lambda_1 \lambda_2$  an  $n$ -th root of unity in  $F_0$ . There are only finitely many such  $\lambda_1$  and  $\lambda_2$  in  $F_0$ , so the center of  $H$  is finite. However, we also know that  $H$  is reductive. Therefore,  $H$  must be semisimple. In light of the inclusions in (15), this implies  $H = R_{F/\mathbb{Q}}SU(B, -)$ .  $\square$

Thus Proposition 5.3 determined the Hodge group under certain conditions when  $[L : E]$  was equal to 1 and Proposition 5.4 determined the Hodge group under certain conditions when  $[L : E]$  was equal to 2. We now determine the Hodge group of a simple polarizable  $\mathbb{Q}$ -Hodge structure  $V$  which is an  $E$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  when  $[L : E]$  is equal to  $n$ , namely when  $\dim_L V$  is equal to 2.

**Proposition 5.5.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  such that there exists a CM field  $E$  such that  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$ . Write  $[E : \mathbb{Q}] = r$  and suppose that the endomorphism algebra  $L$  of  $V$  is of Type IV and  $[L : \mathbb{Q}] = nr$ . Then, if  $B \cong M_2(L^{\mathrm{op}})$  is the centralizer of  $L$  in  $\mathrm{End}_{\mathbb{Q}}(V)$  and  $F$  is the subfield of the center of  $L$  fixed by the Rosati involution, we have:*

$$Hg(V) = R_{F/\mathbb{Q}}SU(B, -).$$

*Proof.* Let  $H = Hg(V)$ . The Lefschetz group of  $V$  is  $R_{F/\mathbb{Q}}U(B, -)$ , so we have:

$$(16) \quad H \subseteq R_{F/\mathbb{Q}}U(B, -),$$

where  $U(B, -)$  is an  $F$ -form of  $GL(2)$ .

However,  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$ . Thus by Lemma 5.2 we have:

$$H \subseteq R_{J/\mathbb{Q}}SU(C, -),$$

where  $J$  is the subfield of  $E$  fixed by complex conjugation and  $C \cong M_{2n}(E^{\mathrm{op}})$  is the centralizer of  $E$  in  $\mathrm{End}_{\mathbb{Q}}(V)$ . Here, the group  $SU(C, -)$  is a  $J$ -form of  $SL(2n)$ .

Now since  $L = [\mathrm{End}_{\mathbb{Q}}(V)]^H$ , the center of  $H$  is contained in  $F_0$ , the center of  $L$ . Hence the center of  $H$  is contained in  $R_{F/\mathbb{Q}}U(B, -) \cap F_0$ . So if  $\lambda_1, \dots, \lambda_n$  are elements of the center of  $Hg(V)$ , all

of the  $\lambda_i$  must have norm 1. Moreover, because  $H$  is contained in  $R_{J/\mathbb{Q}}SU(C, -)$ , considering an embedding of  $R_{F/\mathbb{Q}}U(B, -) \cap F_0$  into  $R_{J/\mathbb{Q}}SU(C, -)$ , we must have the product  $\prod_{i=1}^n \lambda_i$  equal to  $\pm 1$ . There are only finitely many such  $\lambda_i$  in  $F_0$ , so the center of  $H$  is finite. However, we also know that  $H$  is reductive. Therefore,  $H$  must be semisimple.

In light of the inclusion in (16), the semisimplicity of  $H$  implies that we actually have:

$$H \subseteq R_{F/\mathbb{Q}}SU(B, -),$$

where  $SU(B, -)$  is an  $F$ -form of  $SL(2)$ .

Now write  $g = [F : \mathbb{Q}]$  and  $q^2 = [L : F_0]$ . We know that the centralizer of  $H$  in  $\text{End}_{\mathbb{Q}}(V)$  is isomorphic to  $L \otimes_{\mathbb{Q}} \mathbb{C} \cong (M_q(\mathbb{C}))^{2g}$ . Thus we have:

$$(17) \quad H_{\mathbb{C}} \subseteq (SL(2))^{2g}.$$

The representation  $V_{\mathbb{C}}$  of  $(SL(2))^{2g}$  is isomorphic to

$$qW_1 \oplus qW_2 \oplus \cdots \oplus qW_{2g}$$

where  $W_i$  is the representation of the image  $i$ th projection map on  $(SL(2))^{2g}$ . Since each representation  $W_i$  is nonzero, for each  $i$  we have:

$$[W_i \otimes W_i^*]^{H_{\mathbb{C}}} \neq 0.$$

Hence, in the decomposition of  $[V_{\mathbb{C}} \otimes V_{\mathbb{C}}^*]^{H_{\mathbb{C}}}$ , the term

$$\left[ q^2 \bigoplus_{i=1}^{2g} W_i \otimes W_i^* \right]^{H_{\mathbb{C}}}$$

has dimension greater than or equal to  $2gq^2$ , which is equal to  $[L : \mathbb{Q}]$ . But since the endomorphism algebra of  $V$  is  $L$ , we know  $[V_{\mathbb{C}} \otimes V_{\mathbb{C}}^*]^{H_{\mathbb{C}}}$  has dimension equal to  $[L : \mathbb{Q}]$ . Thus, for each  $i$ , the term

$$[W_i \otimes W_i^*]^{H_{\mathbb{C}}}$$

has dimension exactly equal to 1. Therefore, the representation of  $H_{\mathbb{C}}$  on  $W_i$  is irreducible.

But since  $H_{\mathbb{C}}$  is semisimple and is contained in  $(SL(2))^{2g}$ , each  $W_i$  is  $q$  times the standard representation  $U_i$  of  $SL(2)$ . Hence  $H_{\mathbb{C}}$  surjects onto each copy of  $SL(2)$  in the inclusion (17). Hence either  $H_{\mathbb{C}}$  is the entire product  $(SL(2))^{2g}$  or  $H_{\mathbb{C}}$  has a factor which is the graph of an isomorphism  $\alpha$  between factors  $SL(2)$  and  $SL(2)$ . Namely, without loss of generality, if  $H_{\mathbb{C}}$  is not the entire product in (17), then we may write:

$$H_{\mathbb{C}} = \Gamma_{\alpha} \times H',$$

where  $\Gamma_{\alpha}$  is the graph of  $\alpha$  and  $H'$  is some  $\mathbb{C}$ -group. Let  $p_1$  denote projection onto the first factor of  $\Gamma_{\alpha}$  and let  $p_2$  denote projection onto the second factor of  $\Gamma_{\alpha}$ .

Now groups of type  $A_l$  have only inner automorphisms. Hence, the automorphism  $\alpha$  is given by  $\alpha(u) = fuf^{-1}$  for some bijective  $\mathbb{Q}$ -homomorphism  $f : U_i \rightarrow U_j$ . Moreover, note that for any  $h \in H_{\mathbb{C}}$ , we have

$$\begin{aligned} h \cdot f(x) &= p_2(h)(f(x)) \\ &= (\alpha \circ p_1)(h)(f(x)) \\ &= f \circ p_1(h) \circ f^{-1}(f(x)) \\ &= f \circ p_1(h)(x) \\ &= f(h \cdot x) \end{aligned}$$

Hence  $f$  is an  $H_{\mathbb{C}}$ -module isomorphism. This  $f$  thus induces an  $H_{\mathbb{C}}$ -module isomorphism between  $W_i$  and  $W_j$ .

However, the endomorphism algebra  $L \otimes_{\mathbb{Q}} \mathbb{C} = [\text{End}_{\mathbb{C}}(V)]^{H_{\mathbb{C}}}$  of  $V$  splits as  $(M_q(\mathbb{C}))^{2g}$ , and we have  $2g$  different representations  $W_i$ . Thus, there can be no  $H_{\mathbb{C}}$ -module isomorphism between any  $W_i$  and  $W_j$ . Therefore, we must have that  $H_{\mathbb{C}}$  is equal to the entire product  $(SL(2))^{2g}$ .  $\square$

Proposition 5.3, Proposition 5.4, and Proposition 5.5 determine the Hodge group under certain conditions when  $[L : E]$  was equal to 1, 2, and  $n$ . The last case we consider is the case when  $[L : E]$  is equal to  $2n$ , namely when  $\dim_L V$  is equal to 1. In this case, the Hodge structure  $V$  is of CM type, meaning that the Hodge group of  $V$  is a torus. In such a case, even the assumption that  $V$  is an  $E$ -Hodge structure of the desired type is not sufficient to completely limit the possibilities for the Hodge group of  $V$ , although as Proposition 5.6 indicates we do have some restriction on the possible Hodge groups. First let us introduce the following notation.

Consider the torus  $U_L$  given on  $\mathbb{Q}$ -points by

$$U_L(\mathbb{Q}) = \{x \in R_{L/\mathbb{Q}} G_m(\mathbb{Q}) = L^* \mid x\bar{x} = 1\}$$

Now define the torus  $SU_{L/E}$  to be the subtorus of  $U_L$  given on  $\mathbb{Q}$ -points by

$$SU_{L/E}(\mathbb{Q}) = \{x \in U_L(\mathbb{Q}) \mid \text{Norm}_{L/E}(x) = 1\}$$

We then have the following:

**Proposition 5.6.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  such that there exists a CM field  $E$  such that  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$ . Write  $[E : \mathbb{Q}] = r$  and suppose that the endomorphism algebra  $L$  of  $V$  is of Type IV and  $[L : \mathbb{Q}] = 2nr$ . Then we have:*

$$Hg(V) \subseteq SU_{L/E}.$$

*Proof.* The result follows from Lemma 5.2.  $\square$

Thus, the fact of being an  $E$ -Hodge structure with Hodge numbers of the desired form places some restriction on the possible Hodge groups of a simple  $\mathbb{Q}$ -Hodge structure  $V$  with Hodge numbers  $(n, 0, \dots, 0, n)$  when  $V$  has endomorphism algebra of Type IV and  $V$  is of CM type. Namely, we know that the Hodge group of  $V$  must be some subtorus of the torus  $SU_{L/E}$ . However, in order to get greater restrictions on the possibilities for  $Hg(V)$  we need to place greater restrictions on the field  $E$ .

**Proposition 5.7.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  such that there exists an imaginary quadratic field  $E$  such that  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(p, 0, \dots, 0, p)$ , where  $p$  is an odd prime. Suppose that the endomorphism algebra  $L$  of  $V$  satisfies:*

- (1)  $L$  is of Type IV
- (2)  $L/\mathbb{Q}$  is a Galois extension
- (3)  $[L : \mathbb{Q}] = 4p$

*Then we have:*

$$Hg(V) = SU_{L/E}.$$

*Proof.* By Proposition 5.6, we know:

$$Hg(V) \subseteq SU_{L/E}.$$

Observe that the rank of  $SU_{L/E}$  is  $2p - 1$  since  $SU_{L/E}$  has codimension 1 in  $U_L$ .

Following [8] and [12], for  $\Sigma(L) = \{\sigma_1, \dots, \sigma_{4p}\}$  the set of embeddings of  $L$  into  $\mathbb{C}$ , a CM type  $\Theta \subset \Sigma(L)$  is defined by the criterion:

$$\Theta \cup \bar{\Theta} = \Sigma(L).$$

The Hodge structure  $V$  corresponds to a unique CM type  $\Theta$  on  $L$  [9, Section V.C].

Now let  $\tilde{L}$  be the Galois closure of  $L$  and consider  $\text{Gal}(\tilde{L}/\mathbb{Q})$ . For every  $g \in \text{Gal}(\tilde{L}/\mathbb{Q})$  and for every  $\sigma_i \in \Theta$  let  $\sigma_i^g$  be the element of  $\Sigma(L)$  defined by:

$$\begin{aligned}\sigma_i^g : L &\rightarrow \mathbb{C} \\ x &\mapsto g \cdot \sigma_i(x).\end{aligned}$$

Then let  $\Theta^g$  be the CM-type on  $L$  given by:

$$\Theta^g = \{\sigma_1^g, \dots, \sigma_{4p}^g\}.$$

The *Kubota rank* of  $\Theta$ , denoted  $\text{Rank}(\Theta)$ , is the rank over  $\mathbb{Z}$  of the submodule of  $\mathbb{Z}[\Sigma(L)]$  spanned by the set:

$$\{\Theta^g \mid g \in \text{Gal}(L^c/\mathbb{Q})\}.$$

We then have [9, Proposition V.D.5]:

$$(18) \quad \text{Rank}(\Theta) = \dim_{\mathbb{Q}} Hg(V).$$

Tankeev proves in [26, Corollary 3.15], that for a simple CM type  $\Theta$  on a CM field  $L$  of degree  $2p$  over  $\mathbb{Q}$ , where  $p$  is an odd prime, we have:

$$\text{Rank}(\Theta) \geq 2p - 1.$$

Hence it follows from (18) that we have  $\dim_{\mathbb{Q}} Hg(V) \geq 2p - 1$  and so the result is proved.  $\square$

We may then combine the results of Propositions 5.3, 5.4, 5.5, and 5.7 in order to get the following general result about the Hodge group of a simple polarizable  $\mathbb{Q}$ -Hodge structure  $V$  having the property that  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(p, 0, \dots, 0, p)$ , where  $E$  is an imaginary quadratic field.

**Theorem 5.8.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  such that there exists an imaginary quadratic field  $E$  such that  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(p, 0, \dots, 0, p)$ , where  $p$  is a prime. Suppose that the endomorphism algebra  $L$  of  $V$  is of Type IV in Albert's classification. Write  $m = \frac{4p}{[L:\mathbb{Q}]}$ . Let  $B \cong M_m(L^{\text{op}})$  be the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$  and let  $F$  be the subfield of the center of  $L$  fixed by the Rosati involution. Then we have:*

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}} SU(B, ^-) & \text{if } [L:\mathbb{Q}] \neq 4p \\ SU_{L/E} & \text{if } [L:\mathbb{Q}] = 4p \text{ and either } p = 2 \text{ or } L/\mathbb{Q} \text{ is a Galois extension.} \end{cases}$$

Here  $SU_{L/E}$  is the algebraic torus described by:

$$SU_{L/E}(\mathbb{Q}) = \{x \in R_{F/\mathbb{Q}} U(L^{\text{op}}, ^-)(\mathbb{Q}) \mid \text{Norm}_{L/E}(x) = 1\}$$

*Proof.* Observe that since  $\dim_E V = 2p$ , we must have  $[L:E]$  equal to one of 1, 2,  $p$ , or  $2p$ .

The case when  $[L:E] = 1$  is taken care of by Proposition 5.3.

In the case when  $[L:E] = 2$ , by Albert's classification [3, Chapter X, §11] the endomorphism algebra  $L$  is a CM field of degree 4 over  $\mathbb{Q}$ . Now  $L$  induces a decomposition:

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma_i \in \Sigma(L)} (V^{w,0}(\sigma_i) \oplus V^{0,w}(\sigma_i)),$$

where  $V^{w,0}(\sigma_i)$  and  $V^{0,w}(\sigma_i)$  denote the subsets of  $V^{w,0}$  and  $V^{0,w}$ , respectively, where  $L$  acts via  $\sigma_i$ . Now let:

$$n_{\sigma_i} = \dim_{\mathbb{C}} V^{w,0}(\sigma_i) \text{ and } n_{\overline{\sigma_i}} = \dim_{\mathbb{C}} V^{w,0}(\overline{\sigma_i}).$$

Then for all  $i$  we have:

$$n_{\sigma_i} + n_{\overline{\sigma_i}} = p.$$

Hence  $n_{\sigma_i}$  and  $n_{\overline{\sigma_i}}$  are coprime and thus the result follows from Proposition 5.4.

When  $[L:E] = p$ , Albert's classification [3, Chapter X, §11] yields that  $L$  is a CM field of degree  $2p$  over  $\mathbb{Q}$ . Hence the result follows from Proposition 5.5.

Lastly, consider the case when  $[L : E] = 2p$ , namely when  $[L : \mathbb{Q}] = 4p$ . If  $p = 2$ , then Albert's classification [3, Chapter X, §11] yields that either  $L$  is a CM field of degree 8 over  $\mathbb{Q}$  or  $L$  is a division algebra of degree 4 over an imaginary quadratic field  $F_0$ . However, because  $V$  is simple as a  $\mathbb{Q}$ -Hodge structure, Theorem 3.1 in [27] indicates that this latter case cannot occur. So  $L$  must be a CM field of degree 8. Moreover, by Lemma 5.2 because  $V$  is an  $E$ -Hodge structure, we know  $Hg(V)$  is contained in  $SU_{L/E}$ , which has dimension 3. However by Lemma 6.1, the group  $Hg(V)$  has dimension at least 3. So  $Hg(V)$  is equal to  $SU_{L/E}$ .

When  $p$  is an odd prime, then by Albert's classification [3, Chapter X, §11], the endomorphism algebra  $L$  is a CM field of degree  $4p$  over  $\mathbb{Q}$ . Then the result follows by Proposition 5.7. So the result is proved.  $\square$

## 6. A LOWER BOUND ON THE RANK OF $Hg(V)$

The following lemma restates a result proved by Orr [19, Theorem 1.1] for abelian varieties, which generalized an earlier result by Ribet [20] for abelian varieties of CM-type.

**Lemma 6.1.** *Let  $V$  be a polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$ . Suppose the endomorphism algebra  $L$  of  $V$  is commutative. Then the rank as an algebraic group over  $\mathbb{Q}$  of the Hodge group of  $V$  satisfies:*

$$\text{Rank}(Hg(V)) \geq \log_2(2n)$$

*Proof.* Let  $M$  be the Mumford-Tate group of  $V$  and let

$$\rho : M \rightarrow GL(V)$$

be the identical representation of  $M$ . Consider the decomposition

$$M_{\mathbb{R}} = Z_{\mathbb{R}} \cdot M_{1,\mathbb{R}} \cdots M_{s,\mathbb{R}}$$

of  $M_{\mathbb{R}}$ , as the almost direct product of the connected center  $Z_{\mathbb{R}}$  of  $M_{\mathbb{R}}$  and  $\mathbb{R}$ -simple factors  $M_{1,\mathbb{R}}, \dots, M_{s,\mathbb{R}}$ . In fact, these  $M_{i,\mathbb{R}}$  are absolutely simple (see Remark 1.18 in [15]). Because  $L = \text{End}_M(V)$  is commutative, the multiplicities of each such simple factor must be 1.

Now decompose the representation  $\rho \otimes_{\mathbb{Q}} \mathbb{C}$  of  $M_{\mathbb{C}}$  as

$$\chi \boxtimes \rho_1 \boxtimes \cdots \boxtimes \rho_s,$$

where  $\chi$  is a character of  $\text{Lie}(Z)$  and  $\rho_i$  is a representation of  $\text{Lie}(M_i)$ . Let  $\lambda_i$  be the highest weight of  $\rho_i$ . Using [15, Theorem 3.11], all the  $\lambda_i$  have length less than or equal to 1 and thus are minuscule. Since the  $\lambda_i$  are minuscule, the weight spaces of the representation  $V(\lambda_i)$  are all one-dimensional.

Now let  $T$  be a maximal torus in  $M$  and consider the representation  $\rho|_T \otimes_{\mathbb{Q}} \mathbb{C}$ . Let  $r$  be the rank of  $T$  and write

$$T_{\mathbb{C}} \cong \mathbb{G}_{m,1} \times \cdots \times \mathbb{G}_{m,r}.$$

Then without loss of generality, we may suppose that for each  $i$  the unique irreducible component of  $M_{\mathbb{C}}$  containing  $\mathbb{G}_{m,i}$  is  $M_i$  and that  $\rho_i|_{\mathbb{G}_{m,i}}$  describes the corresponding action of  $\mathbb{G}_{m,i}$ .

Since each simple factor in the decomposition of  $M$  has multiplicity 1, we have:

$$(\text{number of weights of } (\rho|_T \otimes_{\mathbb{Q}} \mathbb{C})) = \prod_{i=1}^r (\text{number of weights of } \rho_i|_{\mathbb{G}_{m,i}})$$

Note that because the weight spaces of each  $V(\lambda_i)$  have dimension 1, for  $1 \leq i \leq r$ , we have:

$$\dim_{\mathbb{C}} V(\lambda_i) = \text{number of weights of } \rho_i|_{\mathbb{G}_{m,i}}.$$

Then, since the representation  $V \otimes_{\mathbb{Q}} \mathbb{C}$  given by  $\rho|_T \otimes_{\mathbb{Q}} \mathbb{C}$  decomposes as:

$$\bigotimes_{i=1}^r V(\lambda_i),$$

where the dimension of  $V(\lambda_i)$  is equal to the number of weights of  $\rho_i|_{\mathbb{G}_{m,i}}$ , we have

$$(19) \quad \dim_{\mathbb{C}}(V \otimes_{\mathbb{Q}} \mathbb{C}) = (\text{number of weights of } (\rho|_T \otimes_{\mathbb{Q}} \mathbb{C})).$$

Let  $S$  be the set of  $\text{Aut}(\mathbb{C}/\mathbb{Q})$ -conjugates of the cocharacter  $\mu : \mathbb{G}_m \rightarrow GL_{V,\mathbb{C}}$  which factors through  $M_{\mathbb{C}}$ . Let  $S'$  be the set of cocharacters of  $T_{\mathbb{C}}$  that are  $M(\mathbb{C})$ -conjugate to an element of  $S$ . Note that the images of the  $M(\mathbb{C})$ -conjugates of elements of  $S$  generate  $M_{\mathbb{C}}$ . Moreover, because every cocharacter of  $M$  is  $M(\mathbb{C})$ -conjugate to a cocharacter of  $T_{\mathbb{C}}$ , the images of the  $M(\mathbb{C})$ -conjugates of elements of  $S'$  still generate  $M_{\mathbb{C}}$ . Consider the action of the Weyl group of  $M$  on

$$X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{Hom}(\mathbb{G}_m, T_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Since  $S'$  is closed under this action and  $M_{\mathbb{C}}$  is generated by the images of the  $M(\mathbb{C})$ -conjugates of elements of  $S'$ , it must be the case that  $S'$  spans  $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$  as a  $\mathbb{Q}$ -vector space.

Thus, let  $\Delta$  be a basis for  $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$  contained in  $S'$ . So,  $|\Delta|$  is equal to  $r$ . Any weight  $\lambda \in \text{Hom}(T_{\mathbb{C}}, \mathbb{G}_m)$  of the representation  $\rho|_T \otimes_{\mathbb{Q}} \mathbb{C}$  is then determined by the integers  $\langle \lambda, \delta \rangle$  for  $\delta$  in  $\Delta$ . But any  $\delta$  in  $\Delta$  is also in  $S'$  and thus is some  $\text{Aut}(\mathbb{C}/\mathbb{Q})$ - and  $M(\mathbb{C})$ -conjugate of  $\mu$ . Recall that for any  $z \in \mathbb{C}^*$ , the map  $\rho \circ \mu$  acts as multiplication by  $z^w$  on  $V^{w,0}$  and as the identity on  $V^{0,w}$ . Hence, if  $\delta \in \Delta$ , the integer  $\langle \lambda, \delta \rangle$  can only be 0 or  $w$ . Namely, we have:

$$(\text{number of weights of } (\rho|_T \otimes_{\mathbb{Q}} \mathbb{C})) \leq 2^r.$$

However, this bound may be reduced by noting that  $M$  contains the homotheties. Then there is a unique cocharacter

$$\nu : \mathbb{G}_m \rightarrow M_{\mathbb{C}}$$

such that for  $z \in \mathbb{C}^*$ , the map  $\rho \circ \nu$  acts by multiplication by  $z^w$ . So, in fact,  $\nu$  may be viewed as an element of  $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$  and, moreover,  $\langle \lambda, \nu \rangle$  is equal to  $w$  for any weight  $\lambda$  of  $\rho|_T \otimes_{\mathbb{Q}} \mathbb{C}$ . We may thus choose a new subset  $\Delta'$  of  $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$  and  $S'$  such that  $\nu \cup \Delta'$  forms a basis of  $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Repeating the above arguments for  $\Delta'$  then yields:

$$(20) \quad (\text{number of weights of } (\rho|_T \otimes_{\mathbb{Q}} \mathbb{C})) \leq 2^{r-1}.$$

Combining (19) and (20) yields

$$\log_2(2n) \leq r - 1,$$

where  $r$  is the rank of  $M$  as an algebraic group over  $\mathbb{Q}$ . However we know  $M$  is the almost direct product inside  $GL_V$  of  $\mathbb{G}_{m,\mathbb{Q}}$  and  $Hg(V)$ . Hence, the rank of  $Hg(V)$  is equal to  $r - 1$ . This finishes the proof.  $\square$

## 7. MAIN RESULTS

We now apply the results of Sections 3, 4, 5, and 6 to simple polarizable  $\mathbb{Q}$ -Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$  when  $n$  is equal to 1, a prime  $p$ , 4 and  $2p$ .

In order to determine the Hodge groups of the above Hodge structures, we use the fact that the Hodge group is always contained the Lefschetz group. In fact, we will show that in the possibilities for  $n$  considered, non-Lefschetz Hodge groups occur only when  $n$  is equal to 4 or  $2p$ . For ease of reference, we list in Table 2 the Lefschetz group of a simple polarizable Hodge structure  $V$  with Hodge numbers  $(n, 0, \dots, 0, n)$  depending on the type of the endomorphism algebra of  $V$  in Albert's classification of division algebras with positive involution over a number field [3, Chapter X, §11].

In Table 2, let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(n, 0, \dots, 0, n)$ . Let  $L$  be the endomorphism algebra of  $V$  and let  $F$  be the subfield of the center of  $L$  fixed by the Rosati involution. Write  $2n = m[L : \mathbb{Q}]$ . Then let  $B \cong M_m(L^{\text{op}})$  be the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$ .

TABLE 2. Lefschetz Group of  $V$  for Possible  $L$ 

$L$	Lef( $V$ )	
	Odd Weight	Even Weight
Type I	$R_{F/\mathbb{Q}}Sp({}_F V)$	$R_{F/\mathbb{Q}}SO({}_F V)$
Type II	$R_{F/\mathbb{Q}}Sp(B, -)$	$R_{F/\mathbb{Q}}O^+(B, -)$
Type III	$R_{F/\mathbb{Q}}O^+(B, -)$	$R_{F/\mathbb{Q}}Sp(B, -)$
Type IV	$R_{F/\mathbb{Q}}U(B, -)$	$R_{F/\mathbb{Q}}U(B, -)$

**7.1. Hodge Numbers  $(1, 0, \dots, 0, 1)$ .** We begin by considering the case when  $n$  is equal to 1.

**Proposition 7.1.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  and Hodge numbers  $(1, 0, \dots, 0, 1)$ . Then:*

$$Hg(V) = Lef(V).$$

*Proof.* Let  $H = Hg(V)$  be the Hodge group of  $V$ . Using Albert's classification [3, Chapter X, §11], the endomorphism algebra  $L$  of  $V$  is either  $\mathbb{Q}$  or an imaginary quadratic field. However, a result of Totaro [27, Theorem 3.1] shows that when  $L$  is a totally real field and  $\frac{2n}{[L:\mathbb{Q}]} = 2$ , then any simple polarizable even-weight  $\mathbb{Q}$ -Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$  having endomorphisms by  $L$  actually has endomorphisms in a CM quadratic extension of  $L$ .

Namely, when  $V$  has Hodge numbers  $(1, 0, \dots, 0, 1)$ , then  $L$  is equal to  $\mathbb{Q}$  only when  $V$  has odd weight. In this case, using Table 2, the Lefschetz group of  $V$  is  $SL(2)$ . Hence

$$H \subseteq SL(2).$$

By Remark 2.1, we know  $H$  is nontrivial and by Remark 2.2 we know that  $H$  is semisimple. Hence  $H$  is equal to  $SL(2)$ .

If  $L$  is an imaginary quadratic field, then the Lefschetz group of  $V$  is  $U(L^{\text{op}}, -)$ . So we have:

$$H \subseteq U(L^{\text{op}}, -).$$

However since  $\dim_L(V) = 1$ , the Hodge structure  $V$  is of CM type. Therefore  $H$  must be commutative. Namely, the Hodge group  $H$  is a torus contained in the one-dimensional torus  $U(L^{\text{op}}, -)$ . So  $H$  equals  $U(L^{\text{op}}, -)$ .  $\square$

**7.2. Hodge Numbers  $(p, 0, \dots, 0, p)$ .** We next determine the Hodge group when  $n$  is equal to a prime  $p$ .

**Theorem 7.2.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(p, 0, \dots, 0, p)$ , where  $p$  is prime. Then:*

$$Hg(V) = Lef(V).$$

*Proof.* When  $V$  is of odd weight, the result follows using the equivalence of Remark 3.1 together with results of Ribet [21, Theorems 1,2] and Moonen [15, Example 2.7], from whose proofs the following proof extensively borrows. Thus we may assume that  $V$  is of even weight.

Let  $H$  be the Hodge group  $Hg(V)$  of  $V$ . Assume first that  $p$  is odd. Let  $L$  be the endomorphism algebra of  $V$ , the field  $F_0$  its center, and  $F$  the subfield of  $F_0$  fixed by the Rosati involution on  $L$ . Albert's classification of simple  $\mathbb{Q}$ -algebras with positive involution [3, Chapter X, §11] together with a theorem of Totaro [27, Theorem 3.1] yield the following possibilities for  $L$ :

- (1)  $\mathbb{Q}$  (Type I)
- (2) An imaginary quadratic field (Type IV)

(3) A CM-field of degree  $2p$  over  $\mathbb{Q}$  (Type IV)

First consider Case (1). In this case, by Table 2, the Lefschetz group of  $V$  is  $SO(FV)$ , where  ${}_FV$  denotes  $V$  considered as an  $F$ -vector space. By Proposition 3.2, the group  $H$  is either equal to  $Le f(V)$  or  $R_{L/\mathbb{Q}}SU(2^k)$ , where  $2p = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$ . Thus, to prove the result we must show that this second option cannot occur. To do this, we use the following simple combinatorial argument to show that we cannot have  $2p = \binom{2^k}{2^{k-1}}$  for any  $k \geq 3$ . We know:

$$\binom{2^k}{2^{k-1}} = 2^{2^{k-1}} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2^k - 1)}{(2^{k-1})!}.$$

Moreover, by De Polignac's formula for the prime factorization of  $n!$ , the 2-adic order of the term  $(2^{k-1})!$  is  $2^{k-1} - 1$ . So, after cancellation, we have:

$$(21) \quad \binom{2^k}{2^{k-1}} = 2 \cdot \frac{(\text{product of all odd numbers less than } 2^k)}{(\text{product of some odd numbers all less than } 2^{k-1})}.$$

Using known bounds on the prime-counting function  $\pi$  [22, Corollary 1] yields  $\pi(2^k) - \pi(2^{k-1}) \geq 2$  for  $k \geq 3$ . Namely, the numerator in (21) always contains at least two terms not cancelled by the denominator. Hence we cannot have  $2p = \binom{2^k}{2^{k-1}}$  for any  $k \geq 3$ . Thus, indeed, by Proposition 3.2 when  $L = \mathbb{Q}$ , we have  $H = SO(V)$ , which is the Lefschetz group in this case.

Now consider Case (2). In this case, by Table 2, the Lefschetz group of  $V$  is  $U(B, -)$ , where  $B \cong M_p(L^{\text{op}})$  is the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$ . Letting  $\Sigma(L)$  denote the set of embeddings of  $L$  into  $\mathbb{C}$ , the endomorphism algebra  $L$  induces a decomposition:

$$(22) \quad V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in \Sigma(L)} (V^{w,0}(\sigma) \oplus V^{0,w}(\sigma)),$$

where  $V^{w,0}(\sigma)$  and  $V^{0,w}(\sigma)$  denote the subsets of  $V^{w,0}$  and  $V^{0,w}$ , respectively, where  $L$  acts via  $\sigma$ . Now let:

$$(23) \quad n_{\sigma} = \dim_{\mathbb{C}} V^{w,0}(\sigma) \text{ and } n_{\bar{\sigma}} = \dim_{\mathbb{C}} V^{w,0}(\bar{\sigma}).$$

Then for all  $i$  we have:

$$n_{\sigma} + n_{\bar{\sigma}} = p.$$

Hence  $n_{\sigma}$  and  $n_{\bar{\sigma}}$  are coprime and thus the result follows from Proposition 5.1.

Thus Case (3) is the only remaining case to be addressed for  $p$  odd. In this case, by Table 2 the Lefschetz group of  $V$  is  $R_{F/\mathbb{Q}}U(L^{\text{op}}, -)$ . So we have the inclusion:

$$H \subseteq R_{F/\mathbb{Q}}U(L^{\text{op}}, -).$$

We need to show that this inclusion is actually an equality.

Since  $\dim_L(V) = 1$ , the group  $G = R_{F/\mathbb{Q}}U(L^{\text{op}}, -)$  is the kernel of the natural norm map between  $\mathbb{Q}$ -tori:

$$R_{L/\mathbb{Q}}(\mathbb{G}_m) \rightarrow R_{F/\mathbb{Q}}(\mathbb{G}_m).$$

Since  $[F : \mathbb{Q}] = p$  and hence  $\dim_{\mathbb{Q}}(G) = p$ , in order to show  $H = G$ , we just need to show that  $\dim_{\mathbb{Q}}(H) = p$ .

Consider the character groups  $X^*(H)$ ,  $X^*(G)$ , and  $X^*(R_{L/E}(\mathbb{G}_m))$ . Since both  $H$  and  $G$  are contained in  $R_{L/\mathbb{Q}}(\mathbb{G}_m)$ , the groups  $X^*(H)$  and  $X^*(G)$  are quotients of  $X^*(R_{L/\mathbb{Q}}(\mathbb{G}_m))$ . Here  $X^*(R_{L/E}(\mathbb{G}_m))$  is the free abelian group on the embeddings  $\sigma \in \Sigma(L)$ . The group  $X^*(R_{L/E}(\mathbb{G}_m))$  has a natural left action by  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . The group  $X^*(G)$  is the quotient of  $X^*(R_{L/\mathbb{Q}}(\mathbb{G}_m))$  by the relation  $\sigma + \bar{\sigma} = 0$  for all embeddings  $\sigma \in \Sigma(L)$ . The group  $X^*(H)$  is a quotient of  $X^*(G)$  with the property that the images of the embeddings  $\sigma$  in  $X^*(H)$  are all distinct. Indeed, since  $L = \text{End}_H(V)$  is commutative, the  $H$ -module  $V$  cannot have multiplicities greater than 1 in its decomposition into simple modules over  $H$ , each corresponding to an embedding  $\sigma \in \Sigma(L)$ .



Now, consider the homomorphism

$$\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(X^*(H))$$

giving the action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on  $\text{Aut}(X^*(H))$ . Because the elements of  $X^*(H)$  are the embeddings  $\sigma \in \Sigma(L)$ , each one occurring once, we have:

$$\text{Ker}(\rho) = \text{Gal}(\overline{\mathbb{Q}}/\tilde{L}),$$

where  $\tilde{L}$  denotes the Galois closure of  $L$ . Hence the order of  $\text{Im}(\rho)$  is  $[\tilde{L} : \mathbb{Q}]$ . But  $[\tilde{L} : \mathbb{Q}]$  is divisible by the prime number  $p$ , since  $[L : \mathbb{Q}] = 2p$ . Hence we may choose some  $g \in \text{Im}(\rho)$  of order  $p$ . Since  $g$  is in  $\text{Aut}(X^*(H))$ , the action of  $g$  on the  $\mathbb{Q}$ -vector space

$$Y = X^*(H) \otimes \mathbb{Q}$$

makes  $Y$  into a module over  $\mathbb{Q}[x]/(x^p - 1)$ . We may write  $\mathbb{Q}[x]/(x^p - 1) = \mathbb{Q}(\mu_p) \times \mathbb{Q}$ , where  $\mu_p$  is a  $p$ -th root of unity. Thus write

$$Y = Y_1 \oplus Y_2,$$

where  $Y_1$  is a  $\mathbb{Q}(\mu_p)$ -vector space and  $Y_2$  is a  $\mathbb{Q}$ -vector space. Because  $g$  has order  $p$  and thus does not have order 1, the element  $x$  does not act as the identity on  $Y_1$ . So  $Y_1$  is nonzero. Then the dimension of  $Y_1$  over  $\mathbb{Q}$  is a multiple of  $p - 1$ . Hence we have:

$$\dim_{\mathbb{Q}}(H) \geq p - 1.$$

To show that, in fact,  $\dim_{\mathbb{Q}}(H) = p$  it just remains to show that  $Y_2$  is nonzero. Namely we need to show that  $Y$  contains a nonzero element fixed under the action of  $g$ . Choosing some embedding  $\sigma_0 : L \rightarrow \mathbb{C}$ , it is clear that the element

$$(24) \quad \chi = \sigma_0 + g\sigma_0 + \dots + g^{p-1}\sigma_0$$

in  $X^*(H)$  is fixed by  $g$ . So we just need to show that  $\chi$  is nonzero.

Let  $M$  be the Mumford-Tate group of  $V$  and consider the cocharacter groups  $X_*(H)$ ,  $X_*(M)$ , and  $X_*(R_{L/\mathbb{Q}}(\mathbb{G}_m))$ . The group  $X_*(R_{L/\mathbb{Q}}(\mathbb{G}_m))$  is still the free abelian group on the embeddings  $\sigma \in \Sigma(L)$ . Now, as in (22) consider the decomposition induced by  $L$ :

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in \Sigma(L)} (V^{w,0}(\sigma) \oplus V^{0,w}(\sigma)).$$

Then as in (23), let  $n_{\sigma} = \dim_{\mathbb{C}} V^{w,0}(\sigma)$  and  $n_{\bar{\sigma}} = \dim_{\mathbb{C}} V^{0,w}(\bar{\sigma})$ . Since  $[L : \mathbb{Q}] = 2p$ , we have  $n_{\sigma} + n_{\bar{\sigma}} = 1$ . Thus one of  $n_{\sigma}$ ,  $n_{\bar{\sigma}}$  must be 0 and the other must be 1. Let  $\Theta$  be the set of embeddings acting on  $V^{w,0}$ . So  $\Theta$  determines all of the embeddings  $L \rightarrow \mathbb{C}$  which give the action of  $L \otimes_{\mathbb{Q}} \mathbb{C}$  on  $V \otimes_{\mathbb{Q}} \mathbb{C}$ . If  $\Theta = \{\sigma_1, \dots, \sigma_p\}$ , then view  $\Theta$  as an element of  $X_*(R_{L/\mathbb{Q}}(\mathbb{G}_m))$  by identifying it with  $\sigma_1 + \dots + \sigma_p$  in  $X_*(R_{L/\mathbb{Q}}(\mathbb{G}_m))$ .

Let the Hodge structure  $V$  have corresponding homomorphism

$$h : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \rightarrow GL(V \otimes_{\mathbb{Q}} \mathbb{R}).$$

Let

$$\mu : \mathbb{G}_m \rightarrow R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$$

be the cocharacter such that, for  $z \in \mathbb{C}^*$ , the endomorphism  $h \circ \mu(z)$  acts by multiplication by  $z^{-w}$  on  $V^{w,0}$  and acts as the identity on  $V^{0,w}$ . Then, by definition, the Mumford-Tate group  $M$  is the smallest  $\mathbb{Q}$ -algebraic subgroup of  $GL_V$  such that

$$h \circ \mu : \mathbb{G}_m \rightarrow GL(V \otimes_{\mathbb{Q}} \mathbb{C})$$

factors through  $M_{\mathbb{C}}$ . Thus  $\Theta$  lies in  $X_*(M)$  and, in fact, the  $\mathbb{Z}[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$ -submodule of  $X_*(R_{L/\mathbb{Q}}(\mathbb{G}_m))$  generated by  $\Theta$  is contained in  $X_*(M)$ . But  $X_*(H)$  consists of the elements of  $X_*(M)$  which have

degree 0 in  $X_*(R_{L/\mathbb{Q}}(\mathbb{G}_m))$ . Thus, in particular,  $\eta = \Theta - \overline{\Theta}$  is an element of  $X_*(H)$ . Since each embedding  $\sigma \in \Sigma(L)$  has coefficient  $\pm w$  in  $\eta$ , we have:

$$\langle \eta, \sigma \rangle = \pm w \text{ for all } \sigma \in \Sigma(L),$$

where  $\langle \cdot, \cdot \rangle : X_*(H) \times X^*(H) \rightarrow \mathbb{Z}$  denotes the natural bilinear pairing. Then, from the description of  $\chi$  in (24), the integer  $\langle \eta, \chi \rangle$  is the sum of  $p$  terms each of which is  $\pm w$ . Since  $p$  is odd, this means  $\langle \eta, \chi \rangle$  is nonzero. Hence  $\chi$  must be nonzero. So indeed  $Y$  contains a nonzero element fixed under the action of  $g$  and hence we have  $\dim_{\mathbb{Q}} H = p$ .

Thus, indeed, the result holds whenever  $p$  is odd and so it remains simply to check the result for  $p = 2$ . Namely, when  $V$  is of even weight with Hodge numbers  $(2, 0, \dots, 0, 2)$  we must verify that  $H = G$ . Albert's classification [3, Chapter X, §11] together with [27, Theorem 3.1] yield the following possibilities for  $L$ :

- (1)  $\mathbb{Q}$  (Type I)
- (2) A totally definite quaternion algebra over  $\mathbb{Q}$  (Type III)
- (3) A CM-field of degree 4 (Type IV)

Consider Case (1) first. In this case, the Lefschetz group of  $V$  is  $SO(V)$ . So we know:

$$H \subseteq SO(V),$$

where  $SO(V)$  has rank 2. Observe that by Lemma 6.1 the rank of  $H$  as an algebraic group over  $\mathbb{Q}$  must be greater than or equal to 2. Moreover, by Remark 2.2, the group  $H$  is semisimple. However, it is easy to see that  $SO(4) \cong SL(2) \times SL(2)$  has no semisimple proper subgroups of rank greater than or equal to 2, since  $SL(2)$  has rank 1. So indeed  $H$  is equal to the Lefschetz group  $SO(V)$ .

For Case (2), since  $\dim_L(V) = 1$  we have:

$$Lef(V) = Sp(L^{\text{op}}, -),$$

which is a  $\mathbb{Q}$ -form of  $SL_2$  and hence of rank 1. Thus we must have  $H = Lef(V)$ .

For Case (3), we have

$$Lef(V) = R_{F/\mathbb{Q}}U(L^{\text{op}}, -),$$

which is a torus of dimension 2. Applying Lemma 6.1 yields that  $H$  has dimension at least 2, so we must have  $H = Lef(V)$ .

Thus, indeed, when  $p = 2$  the Hodge group of  $V$  is always equal to the Lefschetz group of  $V$ , which finishes the proof.  $\square$

**7.3. Hodge Numbers**  $(4, 0, \dots, 0, 4)$ . We now determine all the possible Hodge groups when  $n$  is equal to 4. Recall the following notation. If  $V$  is a simple polarizable  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$ , let  $L$  be the endomorphism algebra of  $V$ . Let  $F_0$  be the center of  $L$  and let  $F$  be the subfield of  $F_0$  fixed by the Rosati involution. Now write  $2n = m[L : \mathbb{Q}]$ . Then let  $B \cong M_m(L^{\text{op}})$  be the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$ .

**Theorem 7.3.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(4, 0, \dots, 0, 4)$ . Then the Hodge group  $Hg(V)$  of  $V$  is described by Table 3. In particular, we have:*

$$Hg(V) = Lef(V)$$

except in the following cases:

- (1) If  $L = \mathbb{Q}$  and  $w$  is odd, then we can also have:

$$Hg(V) = SL(2) \times SO(4)$$

and both possible Hodge groups do occur.

(2) If  $L = \mathbb{Q}$  and  $w$  is even, then we can also have:

$$Hg(V) = SO(7)$$

and both possible Hodge groups do occur.

(3) If  $L$  is an imaginary quadratic field such that  $V$  is an  $L$ -Hodge structure with Hodge numbers  $(2, 0, \dots, 0, 2)$ , then:

$$Hg(V) = R_{F/\mathbb{Q}}SU(B, -).$$

(4) If  $L$  is a CM field of degree 8 containing an imaginary quadratic field  $E$  such that  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(2, 0, \dots, 0, 2)$ , then:

$$Hg(V) = SU_{L/E}.$$

Here  $SU_{L/E}$  is the torus defined by the condition

$$SU_{L/E}(\mathbb{Q}) = \{z \in R_{F/\mathbb{Q}}U(L^{\text{op}}, -)(\mathbb{Q}) \mid \text{Norm}_{L/E}(z) = 1\}.$$

TABLE 3. Hodge groups for  $\mathbb{Q}$ -Hodge structures with Hodge numbers  $(4, 0, \dots, 0, 4)$

$L$	$[L : \mathbb{Q}]$	Possible Hodge Groups		Equal to Lefschetz Group?
		Odd Weight	Even Weight	
Type I	1	$Sp(8)$	$SO(8)$	Yes
		$SL(2) \times SO(4)$	-	No
		-	$SO(7)$	No
	2	$R_{F/\mathbb{Q}}Sp(FV)$	$R_{F/\mathbb{Q}}SO(FV)$	Yes
	4	$R_{F/\mathbb{Q}}Sp(FV)$	-	Yes
Type II	4	$Sp(B, -)$	$O^+(B, -)$	Yes
	8	$R_{F/\mathbb{Q}}Sp(L^{\text{op}}, -)$	-	Yes
Type III	4	$O^+(B, -)$	$Sp(B, -)$	Yes
	8	-	$R_{F/\mathbb{Q}}Sp(L^{\text{op}}, -)$	Yes
Type IV	2	$U(B, -)$	$\bar{U}(B, -)$	Yes
		$SU(B, -)$	$SU(B, -)$	No
	4	$R_{F/\mathbb{Q}}U(B, -)$	$R_{F/\mathbb{Q}}U(B, -)$	Yes
	8	$R_{F/\mathbb{Q}}U(L^{\text{op}}, -)$	$R_{F/\mathbb{Q}}U(L^{\text{op}}, -)$	Yes
		$SU_{L/E}$	$SU_{L/E}$	No

*Proof.* By Albert's classification of division algebras with positive involution over a number field [3, Chapter X, §11] together with [27, Theorem 3.1], the endomorphism algebra  $L$  must be one of the following:

- (1)  $\mathbb{Q}$  (Type I)
- (2) A real quadratic field (Type I)
- (3) A totally real field of degree 4 (Type I)
- (4) A quaternion algebra over  $\mathbb{Q}$  (Type II/ Type III)
- (5) A quaternion algebra over  $F$  with  $[F : \mathbb{Q}] = 2$  (Type II if  $w$  odd/ Type III if  $w$  even)
- (6) An imaginary quadratic field (Type IV)
- (7) A CM field of degree 4 (Type IV)
- (8) A CM field of degree 8 (Type IV)

Using Table 2, when  $L$  is of Type I, then the Lefschetz group of  $V$  is  $R_{F/\mathbb{Q}}Sp(FV)$  when  $w$  is odd and  $R_{F/\mathbb{Q}}SO(FV)$  when  $w$  is even. Case (1) is then taken care of by Proposition 3.7, Case (2) is taken care of by Proposition 3.5, and Case (3) is taken care of by Proposition 3.2. From these, we conclude that when  $L$  is of Type I, then the Hodge group of  $V$  is always equal to the Lefschetz

group of  $V$  except when  $L = \mathbb{Q}$ , in which case, the two additional groups  $SL(2) \times SO(4)$  and  $SO(7)$  are also possible. Similarly, using Table 2, Case (4) is taken care of by Proposition 4.2 as well as by Proposition 4.3 and Case (5) is taken care of by 4.1. From these, we conclude that when  $L$  is of Type II or III, then the Hodge group of  $V$  is always equal to the Lefschetz group of  $V$ . This leaves only the cases when  $L$  is of Type IV to consider. The result of the theorem then follows from the following propositions: Proposition 7.4, Proposition 7.5, and Proposition 7.6, which address Cases (6), (7), and (8) respectively.

**Proposition 7.4.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  and Hodge numbers  $(4, 0, \dots, 0, 4)$  such that the endomorphism algebra  $L$  of  $V$  is an imaginary quadratic field. Let  $B \cong M_4(L^{\text{op}})$  be the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$ . For embeddings  $\sigma \in \Sigma(L)$  of  $L$  into  $\mathbb{C}$ , let*

$$n_{\sigma} = \dim V^{w,0}(\sigma) \text{ and } n_{\bar{\sigma}} = \dim V^{w,0}(\bar{\sigma}).$$

*Then either  $\{n_{\sigma}, n_{\bar{\sigma}}\} = \{1, 3\}$  or  $\{n_{\sigma}, n_{\bar{\sigma}}\} = \{2, 2\}$  and we have:*

$$Hg(V) = \begin{cases} U(B, ^{-}) & \text{if } \{n_{\sigma}, n_{\bar{\sigma}}\} = \{1, 3\} \\ SU(B, ^{-}) & \text{if } \{n_{\sigma}, n_{\bar{\sigma}}\} = \{2, 2\}. \end{cases}$$

*Proof.* We know  $n_{\sigma} + n_{\bar{\sigma}} = 4$ , and by [27, Theorem 3.1] we cannot have  $\{n_{\sigma}, n_{\bar{\sigma}}\} = \{0, 4\}$ . Therefore either we have  $\{n_{\sigma}, n_{\bar{\sigma}}\} = \{1, 3\}$  or we have  $\{n_{\sigma}, n_{\bar{\sigma}}\} = \{2, 2\}$ . The case when  $\{n_{\sigma}, n_{\bar{\sigma}}\} = \{1, 3\}$  is taken care of by Proposition 5.1. In the case when  $\{n_{\sigma}, n_{\bar{\sigma}}\} = \{2, 2\}$ , then  $V$  is an  $L$ -Hodge structure with Hodge numbers  $(2, 0, \dots, 0, 2)$ . Hence the result follows from Proposition 5.3.  $\square$

Thus Proposition 7.4 proves the result of Theorem 7.3 in Case (6) of the proof. Proposition 7.5 now deals with Case (7) in the proof of Theorem 7.3.

**Proposition 7.5.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  and Hodge numbers  $(4, 0, \dots, 0, 4)$  such that the endomorphism algebra  $L$  of  $V$  is a CM field of degree 4. Let  $F$  be the subfield of  $L$  fixed by complex conjugation and let  $B \cong M_2(L^{\text{op}})$  be the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$ . Then we have:*

$$Hg(V) = R_{F/\mathbb{Q}}U(B, ^{-}).$$

*Proof.* The following proof borrows from Moonen and Zarhin's proof [14, 7.5] of the analogous result for abelian varieties.

We know that  $L \otimes_{\mathbb{Q}} \mathbb{C}$  is isomorphic to 4 copies of  $\mathbb{C}$  indexed over the set  $\Sigma(L) = \{\sigma_1, \bar{\sigma}_1, \sigma_2, \bar{\sigma}_2\}$  of embeddings of  $L$  into  $\mathbb{C}$ . Thus the action of  $L \otimes_{\mathbb{Q}} \mathbb{C}$  on  $V \otimes_{\mathbb{Q}} \mathbb{C}$  yields a decomposition:

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in \Sigma(L)} V^{w,0}(\sigma) \oplus V^{0,w}(\sigma).$$

Here  $V^{w,0}(\sigma)$  and  $V^{0,w}(\sigma)$  are the subspaces of  $V^{w,0}$  and  $V^{0,w}$  respectively where  $L$  acts via  $\sigma$ . Now for  $i \in \{1, 2\}$  let

$$n_{\sigma_i} = \dim V^{w,0}(\sigma_i) \text{ and } n_{\bar{\sigma}_i} = \dim V^{w,0}(\bar{\sigma}_i).$$

Hence we have  $n_{\sigma_i} + n_{\bar{\sigma}_i} = 2$ . By [27, Theorem 3.1], if either  $n_{\sigma_1}n_{\bar{\sigma}_1} + n_{\sigma_2}n_{\bar{\sigma}_2} = 0$  or  $n_{\sigma_1} = 1 = n_{\sigma_2}$ , then the Hodge structure  $V$  cannot be simple. Hence, without loss of generality, we have

$$(25) \quad (n_{\sigma_1}, n_{\bar{\sigma}_1}) = (2, 0) \text{ and } (n_{\sigma_2}, n_{\bar{\sigma}_2}) = (1, 1)$$

Let  $H = Hg(V)$  be the Hodge group of  $V$ . The Lefschetz group of  $V$  is  $R_{F/\mathbb{Q}}U(B, ^{-})$ . Hence we have:

$$(26) \quad H \subseteq R_{F/\mathbb{Q}}U(B, ^{-}).$$

Let us consider the center  $Z$  of  $H$ . Because of the inclusion in (26), we must have:

$$Z \subseteq R_{F/\mathbb{Q}}U(L^{\text{op}}, ^{-}),$$

where the group  $R_{F/\mathbb{Q}}U(L^{\text{op}}, -)$  is a two-dimensional torus. So  $Z$  is a subtorus of dimension at most 2.

Suppose  $Z$  were zero. Then we would have:

$$H \subseteq R_{F/\mathbb{Q}}SU(B, -).$$

However the centralizer of  $R_{F/\mathbb{Q}}SU(B, -)$  in  $\text{End}_{\mathbb{Q}}(V)$  is a quaternion algebra over  $F$ , whereas the centralizer of  $H$  in  $\text{End}_{\mathbb{Q}}(V)$  is  $L$ . Hence  $Z$  must be nonzero.

Suppose  $Z$  has dimension 1. Then by Lemma 7.3 in [14] there exists an imaginary quadratic subfield  $E$  of  $L$  such that  $Z = SU_{L/E}$ , where  $SU_{L/E}$  is the torus defined by the condition  $SU_{L/E}(\mathbb{Q}) = \{z \in R_{F/\mathbb{Q}}U(L^{\text{op}}, -)(\mathbb{Q}) \mid \text{Norm}_{L/E}(a) = 1\}$ . Now let  $C \cong M_4(E^{\text{op}})$  be the centralizer of  $E$  in  $\text{End}_{\mathbb{Q}}(V)$ . Then having the center  $Z$  of  $H$  equal to  $SU_{L/E}$  implies that we have:

$$H \subseteq SU(C, -).$$

Now suppose  $\nu$  and  $\bar{\nu}$  are the two embeddings of  $E$  into  $\mathbb{C}$ . Then the action of  $E \otimes_{\mathbb{Q}} \mathbb{C}$  on  $V \otimes_{\mathbb{Q}} \mathbb{C}$  yields a decomposition:

$$V \otimes_{\mathbb{Q}} \mathbb{C} = (V^{w,0}(\nu) \oplus V^{0,w}(\nu)) \oplus (V^{w,0}(\bar{\nu}) \oplus V^{0,w}(\bar{\nu})).$$

If  $n_{\nu} = \dim V^{w,0}(\nu)$  and  $n_{\bar{\nu}} = \dim V^{w,0}(\bar{\nu})$ , then we have  $n_{\nu} + n_{\bar{\nu}} = 4$ . Then by Lemma 5.2, we must have  $n_{\nu} = 2 = n_{\bar{\nu}}$ . However, since  $E$  is contained in  $L$ , by (25) we must have  $\{n_{\nu}, n_{\bar{\nu}}\} = \{1, 3\}$ . So  $Z$  cannot have dimension 1.

Hence  $Z$  must have dimension 2. Namely, the center of  $H$  is equal to all of  $R_{F/\mathbb{Q}}U(L^{\text{op}}, -)$ .

Since  $L$  is a CM field of degree 4, we know that  $F$  has degree 2 over  $\mathbb{Q}$ . Let  $\rho_1$  and  $\rho_2$  be the two embeddings of  $F$  into  $\mathbb{C}$ . Without loss of generality we may assume that the embeddings  $\sigma_1$  and  $\bar{\sigma}_1$  in  $\Sigma(L)$  extend  $\rho_1$  and the embeddings  $\sigma_2$  and  $\bar{\sigma}_2$  in  $\Sigma(L)$  extend  $\rho_2$ . Then the action of  $F \otimes_{\mathbb{Q}} \mathbb{C}$  on  $V \otimes_{\mathbb{Q}} \mathbb{C}$  yields a decomposition:

$$V \otimes_{\mathbb{Q}} \mathbb{C} = X_{\rho_1} \oplus X_{\rho_2},$$

where  $X_{\rho_1}$  and  $X_{\rho_2}$  are 4-dimensional  $\mathbb{C}$ -vector spaces.

Now for the fixed polarization  $\langle, \rangle$  of  $V$  let  $\psi : V \times V \rightarrow F$  be the bilinear form such that:

$$\langle, \rangle = \text{Tr}_{\mathbb{Q}}^F \circ \psi.$$

Let  $\psi_{\rho_1}$  and  $\psi_{\rho_2}$  be the restrictions of  $\psi$  to  $X_{\rho_1}$  and  $X_{\rho_2}$  respectively. For any  $v, w \in V$  and any  $f \in L$  we have:

$$\psi(fv, w) = \psi(v, \bar{f}w)$$

and the field  $F$  is fixed under the Rosati involution on  $L$ . Hence  $X_{\rho_1}$  and  $X_{\rho_2}$  are orthogonal with respect to  $\psi$ . This means  $\psi_{\rho_1}$  and  $\psi_{\rho_2}$  are nondegenerate alternating bilinear forms and we have:

$$(27) \quad H_{\mathbb{C}} \subseteq U(X_{\rho_1}, \psi_{\rho_1}) \oplus U(X_{\rho_2}, \psi_{\rho_2}).$$

Hence, if  $H_{\mathbb{C}}^{ss}$  denotes the semisimple part of  $H_{\mathbb{C}}$  we have:

$$(28) \quad H_{\mathbb{C}}^{ss} \subseteq SU(X_{\rho_1}, \psi_{\rho_1}) \oplus SU(X_{\rho_2}, \psi_{\rho_2}).$$

Now observe that for  $i \in \{1, 2\}$  we may write:

$$(29) \quad X_{\rho_i} = V(\sigma_i) \oplus V(\bar{\sigma}_i),$$

where  $V(\sigma_i)$  and  $V(\bar{\sigma}_i)$  correspond to the subspaces of  $V \otimes_{\mathbb{Q}} \mathbb{C}$  where  $L$  acts by  $\sigma_i$  and  $\bar{\sigma}_i$  respectively. Namely,  $V(\sigma_i)$  and  $V(\bar{\sigma}_i)$  are two-dimensional irreducible  $H_{\mathbb{C}}$ -modules.

Combining the information in the decompositions in (27) and (29), we may write the center  $Z_{\mathbb{C}}$  of  $H_{\mathbb{C}}$  in the form:

$$Z_{\mathbb{C}} = \{(z_1 \cdot \text{Id}, -z_1 \cdot \text{Id}, z_2 \cdot \text{Id}, -z_2 \cdot \text{Id}) \mid z_1, z_2 \in \mathbb{C}\} \subset U(X_{\rho_1}, \psi_{\rho_1}) \oplus U(X_{\rho_2}, \psi_{\rho_2}).$$

Because the  $V(\sigma_i)$  and  $V(\bar{\sigma}_i)$  are two-dimensional irreducible  $H_{\mathbb{C}}$ -modules, the above description of  $Z_{\mathbb{C}}$  yields that the projection of  $H_{\mathbb{C}}^{ss}$  onto each factor  $SU(X_{\rho_i}, \psi_{\rho_i})$  must be nonzero. But both

of the  $SU(X_{\rho_i}, \psi_{\rho_i})$  factors are simple. Thus,  $H_{\mathbb{C}}^{ss}$  surjects onto each factor  $SU(X_{\rho_i}, \psi_{\rho_i})$  in the inclusion in (28).

Now suppose  $H_{\mathbb{C}}^{ss}$  does not surject onto all of  $SU(X_{\rho_1}, \psi_{\rho_1}) \oplus SU(X_{\rho_2}, \psi_{\rho_2})$ . Then  $H_{\mathbb{C}}^{ss}$  is the graph  $\Gamma_{\alpha}$  of an isomorphism  $\alpha : SU(X_{\rho_1}, \psi_{\rho_1}) \rightarrow SU(X_{\rho_2}, \psi_{\rho_2})$ . Let  $p_1$  denote projection onto the first factor of  $\Gamma_{\alpha}$  and let  $p_2$  denote projection onto the second factor of  $\Gamma_{\alpha}$ .

Now groups of type  $A_l$  have only inner automorphisms. Hence the automorphism  $\alpha$  is given by  $\alpha(u) = fuf^{-1}$  for some bijective  $\mathbb{Q}$ -homomorphism  $f : X_{\rho_1} \rightarrow X_{\rho_2}$ . Moreover, note that for any  $h \in H_{\mathbb{C}}$ , we have:

$$\begin{aligned} h \cdot f(x) &= p_2(h)(f(x)) \\ &= (\alpha \circ p_1)(h)(f(x)) \\ &= f \circ p_1(h) \circ f^{-1}(f(x)) \\ &= f \circ p_1(h)(x) \\ &= f(h \cdot x) \end{aligned}$$

Hence,  $f$  is an  $H_{\mathbb{C}}$ -module isomorphism. However the endomorphism algebra  $L = [\text{End}_{\mathbb{Q}}(V)]^H$  is commutative. Therefore there can be no such  $H_{\mathbb{C}}$ -module isomorphism  $f$ . Thus, we must have:

$$H_{\mathbb{C}}^{ss} = SU(X_{\rho_1}, \psi_{\rho_1}) \oplus SU(X_{\rho_2}, \psi_{\rho_2}).$$

Since we already showed that the center of  $H_{\mathbb{C}}$  is all of  $U(L^{\text{op}}, -)$ , this proves that, indeed,  $H$  is equal to all of  $R_{F/\mathbb{Q}}U(B, -)$ , which is the Lefschetz group of  $V$ .  $\square$

The last remaining case to deal with in the proof of Theorem 7.3 is Case (8), namely the case when the endomorphism algebra is a CM field of degree 8.

**Proposition 7.6.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  and Hodge numbers  $(4, 0, \dots, 0, 4)$  such that the endomorphism algebra  $L$  of  $V$  is a CM field of degree 8. Let  $F$  be the subfield of  $L$  fixed by complex conjugation. Then:*

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}}U(L^{\text{op}}, -) & \text{if } L \text{ contains an imaginary quadratic field } E \text{ such that } V \text{ is an} \\ & E\text{-Hodge structure with Hodge numbers } (2, 0, \dots, 0, 2) \\ SU_{L/E} & \text{otherwise.} \end{cases}$$

Here  $SU_{L/E}$  is the torus defined by the condition:

$$SU_{L/E}(\mathbb{Q}) = \{z \in R_{F/\mathbb{Q}}U(L^{\text{op}}, -)(\mathbb{Q}) \mid \text{Norm}_{L/E}(a) = 1\}.$$

*Proof.* Let  $H = Hg(V)$  be the Hodge group of  $V$ . Because  $\dim_L(V) = 1$ , the Hodge structure  $V$  is of CM type, namely  $H$  is itself a torus. Now by Table 2, the Lefschetz group of  $V$  is  $R_{F/\mathbb{Q}}U(L^{\text{op}}, -)$ , which is a four-dimensional torus. So  $H$  is a subtorus of  $R_{F/\mathbb{Q}}U(L^{\text{op}}, -)$  of rank at most 4. However by Lemma 6.1, the rank of  $H$  is greater than or equal to  $\log_2(8) = 3$ . So there are only two choices: either  $H$  has rank 3 or  $H$  has rank 4.

Suppose  $L$  contains an imaginary quadratic field  $E$  such that  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(2, 0, \dots, 0, 2)$ . Then by Proposition 5.6, the group  $H$  is contained in the torus  $SU_{L/E}$ , which has rank 3 and thus  $H$  is equal to  $SU_{L/E}$ .

Conversely, if  $H$  has rank 3, then by Lemma 7.3 in [14], there exists an imaginary quadratic field  $E$  in  $L$  such that  $H$  is equal to  $SU_{L/E}$ . Moreover, by Lemma 5.2 this means  $V$  must be an  $E$ -Hodge structure with Hodge numbers  $(2, 0, \dots, 0, 2)$ .

Hence, if  $L$  contains no such field  $E$ , then  $H$  must have rank 4 and hence  $H$  is equal to the Lefschetz group  $R_{F/\mathbb{Q}}U(L^{\text{op}}, -)$ .  $\square$

Proposition 7.4, Proposition 7.5, and Proposition 7.6 thus indeed verify the statement of Theorem 7.3 in Cases (6), (7), and (8) respectively. Since we have previously confirmed the statement of Theorem 7.3 in Cases (1)-(5), this completes the proof of Theorem 7.3.  $\square$

**7.4. Hodge Numbers  $(2p, 0, \dots, 0, 2p)$ .** The last case of a simple polarizable Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$  that we consider is the case when  $n$  is equal to twice an odd prime.

As in the previous sections, recall the following notation. If  $V$  is a simple polarizable  $\mathbb{Q}$ -Hodge structure with Hodge numbers  $(n, 0, \dots, 0, n)$ , let  $L$  be the endomorphism algebra of  $V$ . Let  $F_0$  be the center of  $L$  and let  $F$  be the subfield of  $F_0$  fixed by the Rosati involution. Now write  $2n = m[L : \mathbb{Q}]$ . Then let  $B \cong M_m(L^{\text{op}})$  be the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$ .

**Theorem 7.7.** *Let  $V$  be a simple polarizable  $\mathbb{Q}$ -Hodge structure of weight  $w \geq 1$  with Hodge numbers  $(2p, 0, \dots, 0, 2p)$ , where  $p$  is an odd prime. Then if  $L$  is of Type I, II, or III in Albert's classification we always have:*

$$Hg(V) = \text{Lef}(V).$$

*However, if  $L$  is of Type IV in Albert's classification and  $L$  contains an imaginary quadratic field  $E$  such that  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(p, 0, \dots, 0, p)$ , then:*

$$Hg(V) = \begin{cases} R_{F/\mathbb{Q}}SU(B, -) & \text{if } [L : \mathbb{Q}] \neq 4p \\ SU_{L/E} & \text{if } [L : \mathbb{Q}] = 4p \text{ and } L/\mathbb{Q} \text{ is a Galois extension.} \end{cases}$$

Here  $SU_{L/E}$  is the algebraic torus described by

$$SU_{L/E}(\mathbb{Q}) = \{x \in R_{F/\mathbb{Q}}U(L^{\text{op}}, -)(\mathbb{Q}) \mid \text{Norm}_{L/E}(x) = 1\}.$$

*Proof.* Let  $H$  be the Hodge group  $Hg(V)$  of  $V$ . Then, using Albert's classification of division algebras with positive involution over a number field [3, Chapter X, §11] together with [27, Theorem 3.1], the possibilities for  $L$  are as follows:

- (1)  $\mathbb{Q}$  (Type I)
- (2) A real quadratic field (Type I)
- (3) A totally real field of degree  $p$  (Type I)
- (4) A totally real field of degree  $2p$  (Type I)
- (5) A quaternion algebra over  $\mathbb{Q}$  (Type II/ Type III)
- (6) A quaternion algebra over  $F$ , where  $[F : \mathbb{Q}] = p$  (Type II if  $w$  odd/ Type III if  $w$  even)
- (7) A CM field of degree 2 (Type IV)
- (8) A CM field of degree 4 (Type IV)
- (9) A CM field of degree  $2p$  (Type IV)
- (10) A CM field of degree  $4p$  (Type IV)

First consider the cases when  $L$  is of Type I. Then by Table 2, the Lefschetz group of  $V$  is  $R_{F/\mathbb{Q}}Sp(FV)$  when  $w$  is odd and  $R_{F/\mathbb{Q}}SO(FV)$  when  $w$  is even. So consider Case (1). By Proposition 3.6, the group  $H$  is either equal to  $\text{Lef}(V)$  or, when  $w$  is even, we may also have  $R_{L/\mathbb{Q}}SU(2^k)$ , where  $4p = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$ . Thus we just need to verify that this second option cannot occur. We use a simple combinatorial argument, as in the proof of Theorem 7.2, to show that  $4p$  is not of the form  $\binom{2^k}{2^{k-1}}$  for any  $k \geq 3$ . We know:

$$\binom{2^k}{2^{k-1}} = 2^{2^{k-1}} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2^k - 1)}{(2^{k-1})!}.$$

Moreover, by De Polignac's formula for the prime factorization of  $n!$ , the 2-adic order of the term  $(2^{k-1})!$  is  $2^{k-1} - 1$ . This means that  $\binom{2^k}{2^{k-1}}$  is congruent to 2 mod 4, and thus  $4p$  cannot be of the form  $\binom{2^k}{2^{k-1}}$  for any  $k \geq 3$ .

Now consider Case (2). By Proposition 3.2, the group  $H$  is either equal to  $\text{Lef}(V)$  or, when  $w$  is even, we may also have  $R_{L/\mathbb{Q}}SU(2^k)$ , where  $2p = \binom{2^k}{2^{k-1}}$  for some  $k \geq 3$ . However we verified

combinatorially in the proof of Theorem 7.2 that we cannot have  $2p = \binom{2^k}{2^{k-1}}$  for any  $k \geq 3$ , hence this latter case cannot occur.

Case (3) is taken care of by Proposition 3.5.

Case (4) is taken care of by Proposition 3.2 since in this case  $\frac{n}{[L:\mathbb{Q}]} = 1$  and thus in both even and odd weights the only possibility for  $H$  is  $\text{Lef}(V)$ . This finishes the cases for  $L$  of Type I.

For  $L$  of Type II or Type III, referring to Table 2, Case (5) is taken care of by Proposition 4.1, again using that  $2p$  cannot be of the form  $\binom{2^k}{2^{k-1}}$  for  $k \geq 3$ .

Case (6) is also taken care of by Proposition 4.1.

By Table 2, the Lefschetz group when  $L$  is of Type IV is  $R_{F/\mathbb{Q}}U(B, -)$ . Hence the cases involving  $L$  of Type IV, namely Cases (7) through (10) all follow from Theorem 5.8.  $\square$

## 8. APPLICATIONS TO THE HODGE CONJECTURE

In Section 7, we determined the possible Hodge groups of simple polarizable  $\mathbb{Q}$ -Hodge structures with Hodge numbers  $(n, 0, \dots, 0, n)$  when  $n$  was equal to 1, a prime  $p$ , 4 and  $2p$ . The results for  $n = 1$ ,  $p$ , and 4 are generalizations of previous results in [21] and [14] about the possible Hodge groups of simple  $n$ -dimensional abelian varieties. However, the results in Section 7 about the Hodge groups when  $n$  is equal  $2p$  are new. Since, by Remark 3.1, there is a polarization-preserving equivalence of categories between the category of  $\mathbb{Q}$ -Hodge structures of odd weight and Hodge numbers  $(n, 0, \dots, 0, n)$ , and the category of complex abelian varieties of dimension  $n$ , it is natural to ask about the implications of Theorem 7.7 for complex abelian varieties. In particular, it is natural to wonder about the implications in terms of both the Hodge Conjecture and the General Hodge conjecture for these complex abelian varieties of dimension  $2p$ .

In order to simplify notation, in the case of an abelian variety  $A$ , we will denote by  $Hg(A)$  and  $\text{Lef}(A)$  the Hodge and Lefschetz groups respectively of the Hodge structure  $V = H^1(A, \mathbb{Q})$ .

If  $A$  has dimension  $n$ , let  $W(A)$  be the set of CM fields  $E$  such that  $V$  is an  $E$ -Hodge structure with Hodge numbers  $(\frac{n}{[E:\mathbb{Q}]}, 0, \dots, 0, \frac{n}{[E:\mathbb{Q}]})$ .

**Corollary 8.1.** *Let  $A$  be a simple abelian variety of dimension  $2p$ , where  $p$  is an odd prime. Suppose the endomorphism algebra  $L$  of the corresponding Hodge structure  $V = H^1(X, \mathbb{Q})$  satisfies either:*

- (1)  *$L$  is of I, II, or III in Albert's classification*
- (2)  *$L$  is of Type IV, there exists an imaginary quadratic field  $E \in W(A)$ , and if  $[L : \mathbb{Q}] = 4p$ , then  $L/\mathbb{Q}$  is Galois*

*Then if the Hodge conjecture is true for all powers of  $A$ , then the General Hodge Conjecture is true for all powers of  $A$ .*

*Proof.* Suppose first that  $L$  satisfies the property that if  $L$  is of Type IV, then  $[L : \mathbb{Q}]$  is not equal to  $4p$ . By Theorem 7.7, the Hodge group  $Hg(A)$  is semisimple and is equal to the semisimple part  $\text{Lef}(A)^{ss}$  of the Lefschetz group of  $A$ . Moreover, when  $L$  is of Type III, then by Shimura's classification of the possible endomorphism algebras of a simple abelian variety [25, Theorem 5], we must have  $L$  a quaternion algebra over  $\mathbb{Q}$ . Namely  $\frac{4p}{[L:\mathbb{Q}]}$  is equal to  $p$ , where  $p$  is odd. So  $A$  satisfies the following two conditions:

- (1)  $Hg(A) = \text{Lef}(A)^{ss}$
- (2) If  $L$  is of Type III, then  $\frac{2\dim A}{[L:\mathbb{Q}]}$  is odd

These conditions are exactly the hypotheses of a result of Abdulali [1, Theorem 5.1], which then shows that under the above circumstances, the Hodge Conjecture for all powers of  $A$  implies the General Hodge Conjecture for all powers of  $A$ .

Hence we need only address the case when  $L$  is of Type IV and  $[L : \mathbb{Q}]$  is equal to  $4p$ . In this case, the abelian variety  $A$  is of CM-type. Hazama showed in [10, Theorem 8.3] and Abdulali



independently showed in [2] that for an abelian variety of CM-type, the Hodge Conjecture implies the General Hodge Conjecture.  $\square$

Thus for abelian varieties of the kind considered in Theorem 7.7, the validity of the Hodge Conjecture implies the validity of the General Hodge Conjecture. We now show that, in fact, for these abelian varieties, the Hodge Conjecture and thus consequently also the General Hodge Conjecture hold. For this, we first introduce a few definitions and some notation.

For any  $E \in W(A)$ , let  $J$  be the maximal totally real subfield of  $E$  and let  $C$  be the centralizer of  $E$  in  $\text{End}_{\mathbb{Q}}(V)$ . The Hodge structure  $V$  may be viewed as an  $E$ -vector space, say of dimension  $l$ . Thus define

$$W_E = \bigwedge_E^l V.$$

Since  $V$  is an  $E$ -Hodge structure, a result of Moonen and Zarhin [13, Section 6] shows that  $W_E$  consists entirely of Hodge classes. The elements of  $W_E$  are called *Weil classes*.

We now introduce the following group  $M(A)$ , originally defined by Murty in [18, Section 3.6.4].

**Definition 8.2.** For a simple abelian variety  $A$ , let the *Murty group*  $M(A)$  be:

$$M(A) = \text{Lef}(A) \cap \left( \bigcap_{E \in W(A)} R_{J/\mathbb{Q}} SU(C, ^-) \right).$$

Thus for any simple abelian variety  $A$ , we have:

$$(30) \quad Hg(A) \subseteq M(A) \subseteq \text{Lef}(A).$$

In [18], Murty proves the following property about the Murty group:

**Proposition 8.3.** [18, Proposition 3.8] *For an abelian variety  $A$ , the Hodge ring*

$$\mathcal{B}^\bullet(A^k) = \bigoplus_{l \geq 0} \left( H^{2l}(A^k, \mathbb{Q}) \cap H^{l,l} \right)$$

*is generated by divisors and Weil classes for all  $k \geq 1$  if and only if*

$$Hg(A) = M(A).$$

We use this property of the Murty group in order to prove the following.

**Corollary 8.4.** *Let  $A$  be a simple abelian variety of dimension  $2p$ , where  $p$  is an odd prime. Suppose the endomorphism algebra  $L$  of the corresponding Hodge structure  $V = H^1(X, \mathbb{Q})$  satisfies either:*

- (1)  *$L$  is of Type I, II, or III in Albert's classification*
- (2)  *$L$  is of Type IV, there exists an imaginary quadratic field  $E \in W(A)$ , and if  $[L : \mathbb{Q}] = 4p$ , then  $L/\mathbb{Q}$  is Galois*

*Then for every  $k \geq 1$ , the Hodge ring  $\mathcal{B}^\bullet(A^k) = \bigoplus_{l \geq 0} (H^{2l}(A^k, \mathbb{Q}) \cap H^{l,l})$  is generated by divisors. Therefore both the Hodge and General Hodge Conjectures are satisfied for every power of  $A$ .*

*Proof.* When  $L$  is of Type I, II, or III in Albert's classification, then by [18, Proposition 3.6]:

$$\text{Lef}(A) = M(A).$$

Since  $A$  corresponds to a simple polarizable Hodge structure  $V$  with Hodge numbers  $(2p, 2p)$ , when  $L$  is of Type I, II, or III, then by Theorem 7.7, we have:

$$Hg(A) = \text{Lef}(A).$$

Hence, in Case (1) in the statement of the corollary, the Hodge group of  $A$  is indeed equal to the Murty group of  $A$ .

In the situation of Case (2) in the statement of the corollary, we have by Theorem 7.7:

$$Hg(A) = \begin{cases} R_{F/\mathbb{Q}}SU(B, ^-) & \text{if } [L : \mathbb{Q}] \neq 4p \\ SU_{L/E} & \text{if } [L : \mathbb{Q}] = 4p, \end{cases}$$

where  $B$  is the centralizer of  $L$  in  $\text{End}_{\mathbb{Q}}(V)$ . Moreover, in this case, the Lefschetz group of  $A$  is:

$$Lef(A) = \begin{cases} R_{F/\mathbb{Q}}U(B, ^-) & \text{if } [L : \mathbb{Q}] \neq 4p \\ U_{L/E} & \text{if } [L : \mathbb{Q}] = 4p, \end{cases}$$

However, observe that this means for some  $E \in W(A)$  with totally real subfield  $J$  and centralizer  $C$  in  $\text{End}_{\mathbb{Q}}(V)$ , we have:

$$Hg(A) = Lef(A) \cap R_{J/\mathbb{Q}}SU(C, ^-).$$

Namely,

$$M(A) \subseteq Hg(A).$$

By (30) above, we thus have  $M(A) = Hg(A)$ . Then Murty's result, Proposition 8.3, implies that for every  $k \geq 1$  the Hodge ring  $\mathcal{B}^\bullet(A^k)$  is generated by divisors and Weil classes.

However in [13, Section 13], Moonen and Zarhin give explicit criteria for when there exist  $E \in W(A)$  such that  $W_E$  contains exceptional Hodge classes, namely Weil classes that are not divisor classes. For a simple abelian variety  $A$ , let  $L$  be its endomorphism algebra,  $F_0$  the center of  $L$ , and  $F$  the maximal totally real subfield of  $F_0$ . Then write  $q^2 = [L : F_0]$ . Moonen and Zarhin show that  $W_E$  contains exceptional Hodge classes precisely in the following cases:

- (1)  $L$  is of Type III and  $E \subsetneq F_0$
- (2)  $L$  is of Type IV,  $q = 1$ , and  $E \subsetneq F$
- (3)  $L$  is of Type IV,  $q \geq 2$  and the composition  $F_{0,-} \hookrightarrow \text{End}_E(V) \xrightarrow{Tr_E} E$  is nonzero

However, under the hypotheses of the Corollary, we know by Shimura's classification of the possible endomorphism algebras of a simple abelian variety [25, Theorem 5] that if  $L$  is of Type III, then  $L$  is a quaternion algebra over  $\mathbb{Q}$  and when  $L$  is of Type IV, then  $L$  is a CM field of degree 2, 4,  $p$ , or  $2p$ . Namely, we always have  $q = 1$ . Moreover, in the Type III case, we have  $F_0 = \mathbb{Q}$ . Hence none of the cases for the existence of exceptional Hodge classes are possible. Therefore, all of the Weil classes in  $W_E$  are actually just divisor classes. So, Murty's Proposition 8.3 implies that for every  $k \geq 1$  the Hodge ring  $\mathcal{B}^\bullet(A^k)$  is generated by divisors.

The Hodge ring being generated by divisors means that the Hodge Conjecture is satisfied for every power of  $A$ . However, by Corollary 8.1, since the Hodge Conjecture holds for all powers of  $A$ , the General Hodge Conjecture holds for all powers of  $A$ .  $\square$

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